webpage:
www.wtworden.org/teaching/topology_541/

book: Purcell — Hyp. Geom. + Knot Theory
+ occassional material from Benedetti + Petronio,
Thurston, and maybe others

Will cover: (see syllabus)

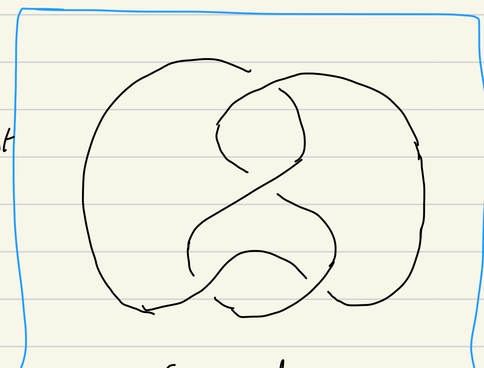
homework: occasional to weekly, but not always collected.

Attendance: yes

Exams: no

Questions/
Requests: Is there anything that someone would really like to see covered?

Where to Begin? 1977



In '77 Thurston showed that fig-8 knot was hyp. by giving explicit construction as gluing of ideal hyp. tets.

Actually, Riley had already proved this by finding a rep $\hookrightarrow PSL_2\mathbb{C}$, and encouraged Thurston to think about hyp. structures on knots $\rightsquigarrow$ fig-8 decomp.

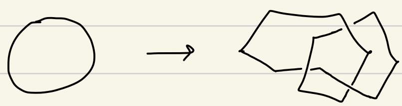
Ch. 1. Decomposition of the figure-8 Knot

Def-n: • A Knot $K \subseteq S^3$ is a subset homeomorphic to S^1 via a piecewise linear homeo-sm.

- Alternatively, can think of K as a PL embedding $K : S^1 \to S^3$, and by abuse of notation identify $K = K(S^1)$.

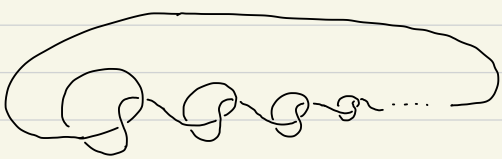
• A link $L \subseteq S^3$ is the image $L(X)$ of a PL embedding $L : X = S^1 \sqcup \ldots \sqcup S^1 \to S^3$ of a disjoint union of copies of S^1 into S^3.

• PL homeo-sm of S^1: takes S^1 to fin many linear segments



• Rmk: Replace PL homeo-sm with smooth diffeo-sm, get same theory. If $K : S^1 \to S^3$ is an embedding that is not PL, then K is a wild Knot.

e.g.



Def-n: Two knots (or links) K_1, K_2 are equivalent if they are ambient isotopic, i.e., $\exists$ a (PL or smooth) homotopy

$$h : S^3 \times [0,1] \longrightarrow S^3$$

restriction $h_t|_{K_1}$ is PL / smooth

s.t. $h(\cdot, t) = h_t : S^3 \longrightarrow S^3$ is a homeo-sm for each t, and $h(K_1, 0) = h_0(K_1) = K_1$

$$h(K_1, 1) = h_1(K_1) = K_2$$

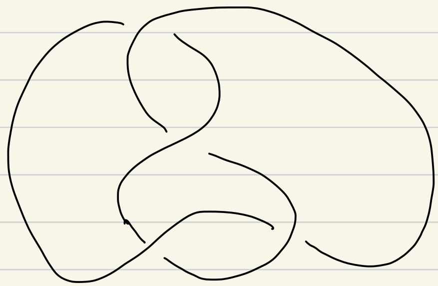
Def-n. Let $K \subseteq S^3$ be a knot (resp. link) in S^3.

- Knot exterior : $S^3 \setminus N(K)$, $N(K) \cong \mathrm{int}(K \times D^2)$ ↳ embedded
  - ↳ compact 3-mfld w/ ∂ homeo-c to T^2
- Knot Complement : $S^3 \setminus K$
  - ↳ open 3-mfld
- define link exteriors & complements similarly.

Def-n: A knot (or link) diagram is a 4-valent graph with over/under crossing info at vertices, embedded in a projection plane $S^2 \subseteq S^3$

[ideal polyhedral decomp: ideal polyhedra P_1, P_2
a map $\varphi : P_1 \sqcup P_2 \longrightarrow M$
$\varphi|_{\mathrm{int}(\Delta)}$ is a homeo-sm for any 1, 2, or 3-cell Δ.

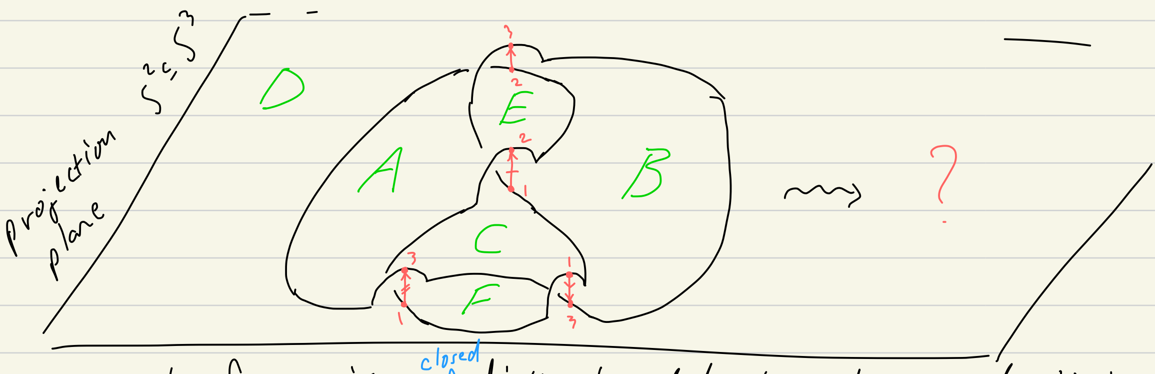
Goal: knot/link diagram $\rightsquigarrow$ decomp. of $S^3 \setminus K$



Def'n: <u>polyhedron</u>: closed B^3 with ∂B^3 labelled by a finite graph.

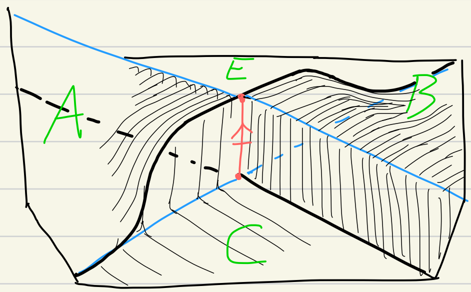
<u>ideal polyhedron</u>: polyhedron $\setminus$ {vertices}

More specific goal: cut $S^3 \setminus K$ into two ideal polyhedra.



- each face is a closed disk, bounded by edges and knot strands. Let $X = A \cup B \cup C \cup D \cup E \cup F$.
- Is X homeo-c to S^2 ?

Near a crossing:

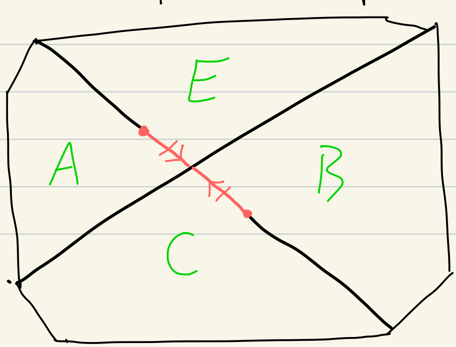


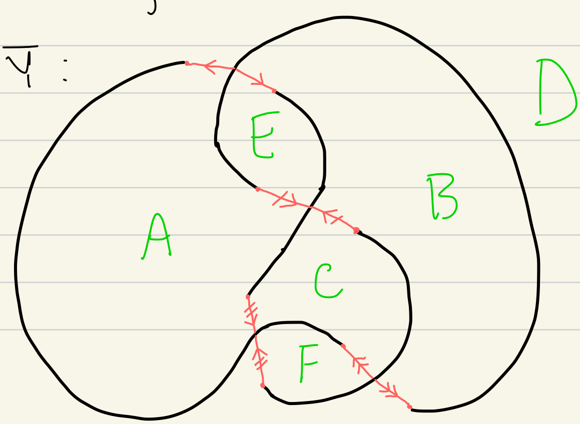
Use paper / 3D model.

- X is pinched at edges (not $\cong S^2$)
  not even a mfld.

- consider the component Y of $S^3 \setminus X$ that lies above the board (the one we are in).
  - Y is an open ball
  - $\overline{Y} = Y \cup X$ is a closed ball which has been pinched at crossings.

If we "unpinch" $\overline{Y}$:





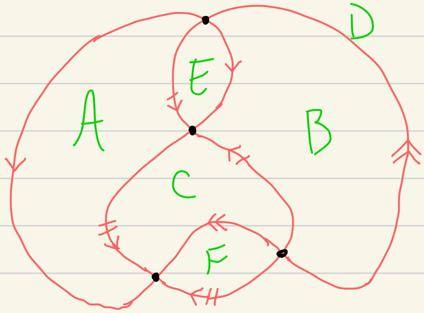
Next: Shrink knot strands to vertices

- recall that we are trying to decompose $S^3 \setminus K$ into ideal polyhedra.

Since
$$\overline{Y}_{\text{unpinched}} \setminus \{\text{strand}\} \cong \overline{Y}_{\text{unpinched}} \setminus \{pt.\}$$

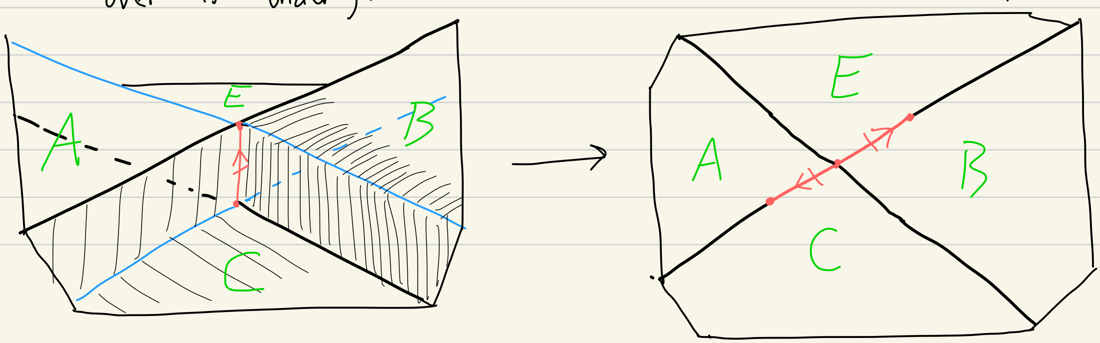
shrinking strands to (ideal) vertices does not change $\overline{Y}_{\text{unpinched}} \setminus K$ (up to homeo-sm).

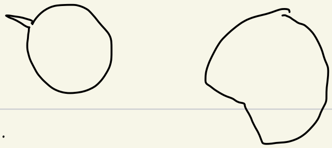
$P_1 \cong \overline{Y}_{\text{unpinched}} \setminus K$



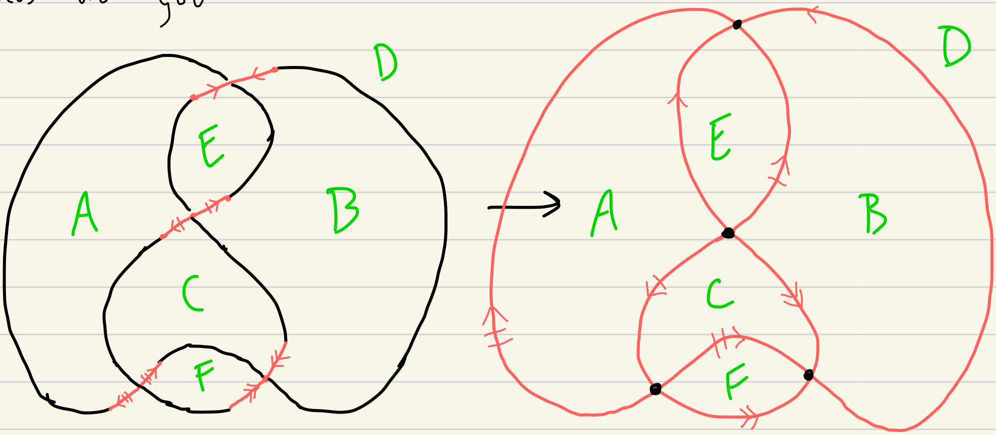
Rmk: this is a view of P_1 from inside.

For the other polyhedron, the process is similar.
- when we unpinch, we should break the knot at the overstrand, instead of the understrand (from inside, over is under):





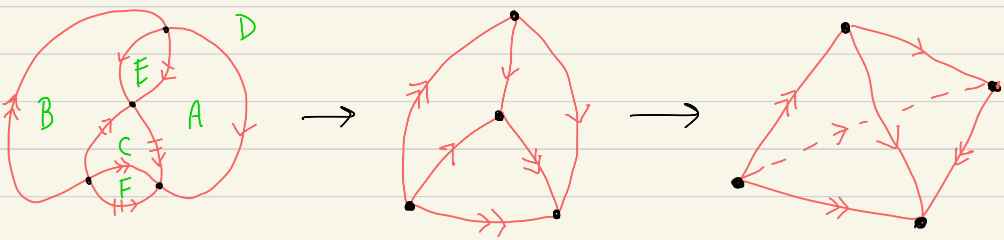
Thus we get:

$P_2 =$ 

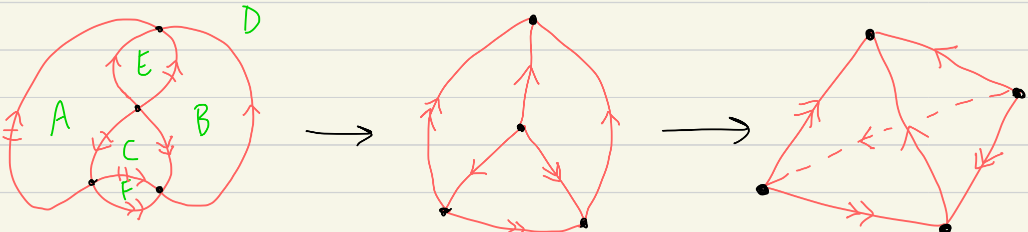
Note: • we are looking at P_2 from the <u>outside</u>.

• P_1 and P_2 have bigon faces. We can collapse these by identifying the two edges of each bigon:

P_1: from outside now



P_2:



(compare to Thurston p39 notes p4)

HW:

Ex. 1.1, 1.2, 1.4

Ex: try to understand Figure 1.26 on page 41 of Thurston's book

Ch 2: Calculating in hyperbolic space

2.0: Digression: models and isometries (B+P ch A.)

- Models for $\mathbb{H}^n$ (think $n = 2, 3$)

(1) Hyperboloid/Minkowski model

$f: \mathbb{R}^{n+1} \to \mathbb{R}$

$x \mapsto \langle x|x\rangle$

$M^n \subseteq f^{-1}(-1)$

$df = (2x_1, \dots, 2x_n, -2x_{n+1})$

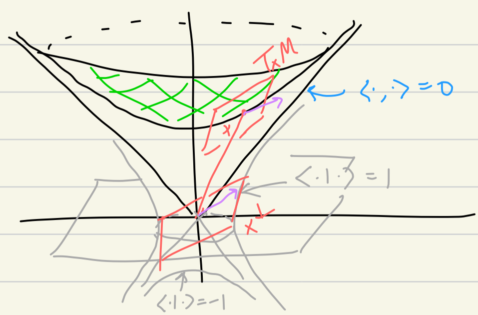
Consider the sym. bilinear form of signature $(n,1)$

$$\langle x|y\rangle_{(n,1)} = x_1y_1 + \dots + x_ny_n - x_{n+1}y_{n+1} \quad \text{on } \mathbb{R}^{n+1}$$

Define: $M^n = \{ x \in \mathbb{R}^{n+1} : \langle x|x\rangle_{(n,1)} = -1, \ x_{n+1} > 0 \}$

(M^n is (component of) the sphere of radius -1, sort of..)

- M^n is a diff-ble oriented hypersurface in $\mathbb{R}^{n+1}$ (pre-image of a regular value of a diff-ble function).



- For $x \in M^n$, we have

$$T_xM^n = \{ y \in \mathbb{R}^{n+1} : \langle x|y\rangle_{(n,1)} = 0 \} = x^\perp \quad \text{(exercise)}$$

Since $\left(\langle x|x\rangle_{(n,1)} = -1, \ x_{n+1} > 0\right) \Rightarrow x \in M^n$

the restriction of $\langle\cdot|\cdot\rangle_{(n,1)}$ to $x^\perp$ is positive definite

($y \in x^\perp$ and $\langle y|y\rangle_{(2,1)} < 0 \Rightarrow \langle \alpha y|\alpha y\rangle = -1$ for some $\alpha > 0 \Rightarrow \alpha y \in M^n$, impossible).

$\langle x-\alpha y, x-\alpha y\rangle$

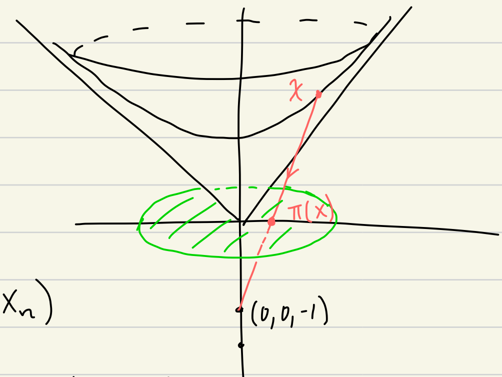
$\langle x,x\rangle - 2\langle \alpha y, x\rangle + \langle \alpha y, \alpha y\rangle = -2$

• Thus we get a Riemannian metric on M^n.
Denote by $\mathbb{M}^n$ the mfld M^n with this metric.

(2) Poincaré (disk) ball model

Let D^n be the open unit ball in $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$

Let $\pi : \mathbb{M}^n \to D^n$ be the restriction of the stereographic projection



$$\pi(x) = \frac{1}{1+x_{n+1}}(x_1, \ldots, x_n)$$

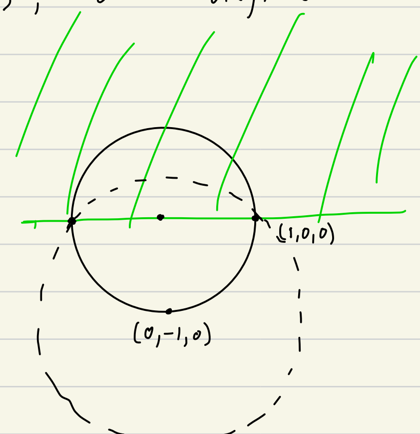
denote by $\mathbb{D}^n$ the manifold D^n with the pull-back metric w.r.t. π^{-1}.

(3) Upper half-(plane) space model:

Let $U^n = \{ x \in \mathbb{R}^n : x_n > 0 \}$, and define

$i : \mathbb{D}^n \to U^n$ by

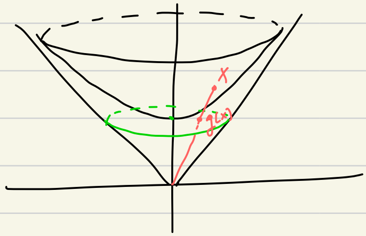
$$x \longmapsto 2\frac{x+e_n}{\|x+e_n\|^2} - e_n$$



Denote by $\mathbb{U}^n$ the upper half-plane U^n with the pull-back metric w.r.t. i^{-1}.

note: i is a sphere inversion in the sphere centered at $-e_n$ of radius $\sqrt{2}$.

(4) Klein Model



Let K^n be the open disk

$$K^n = \{x \in \mathbb{R}^{n+1} : \|x\| < 1,\ x_{n+1} = 1\}$$

and let $f : \mathbb{M}^n \to K^n$ be defined by $x \longmapsto \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1\right)$

$\mathbb{K}^n$ is the Klein model when given the pull-back metric w.r.t. f^{-1}

Rmk: ($\mathbb{K}^n$ is a convex domain in projective space $\mathbb{RP}^n$, so $\mathbb{H}^n$ is a convex projective domain)

Isometries.

For a Riemannian mfld N:

notation: $\mathrm{Isom}(N) = \{\text{isometry group of } N\}$
$\mathrm{Isom}^+(N) = \{\text{orientation preserving subgroup}\}$

Recall: $f : N \to N$ is an isometry means

$$\langle df_x(v) | df_x(w) \rangle_{f(x)} = \langle v | w \rangle_x \quad \forall x \in N,\ v, w \in T_x N$$

Also: Isometries a determined <u>locally</u>, i.e.,

If $f : N \to N$ and $g : N \to N$ are isometries, and $f(y) = g(y)$ and $df_y = dg_y$ for some $y \in N$, then $f = g$

· Let $n = p+q$, and $V \cong \mathbb{R}^n$

$$O(p,q) = O(V, \langle \cdot | \cdot \rangle_{(p,q)}) = \{A \in GL(n,\mathbb{R}) : \langle Ax | Ay \rangle_{(p,q)} = \langle x, y \rangle_{(p,q)} \ \forall \ x,y \in V\}$$

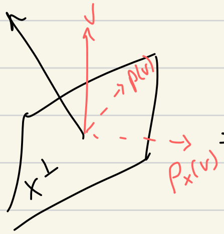
where $\langle e_i, e_j \rangle_{(p,q)} = \begin{cases} 0 & i \neq j \\ 1 & i = j \leq p \\ -1 & i = j > p \end{cases}$

· Let $x \in V$, $\langle x, x \rangle \neq 0$, and let $\text{proj}: V \longrightarrow x^\perp$ be the projection onto $x^\perp$

Let $\rho_x : v \longmapsto 2\,\text{proj}(v) - v$ be the reflection across $x^\perp$

$\rho_x \in O(p,q)$: $\quad \langle 2p(v) - v, 2p(w) - w \rangle$

$$= \langle 2p(v), 2p(w) \rangle - \langle v, 2p(w) \rangle - \langle w, 2p(v) \rangle + \langle v, w \rangle$$



$$= 2\langle p(v), p(w) \rangle - 2\langle p(v), w \rangle$$

$$2\langle p(w), p(v) \rangle - 2\langle p(w), v \rangle + \langle v, w \rangle$$

$$= -2\underbrace{\langle p(v), w - p(w) \rangle}_{=0} - 2\underbrace{\langle p(w), v - p(v) \rangle}_{=0} + \langle v, w \rangle$$

since $p(v) \in V$ and $w - p(w) \in V^\perp$ $\qquad$ ''

$$= \langle v, w \rangle$$

Lemma: $O(p,q)$ is generated by $\{\rho_x : x \in V, \langle x,x\rangle \neq 0\}$

proof: first note that $-I_n = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ is generated by reflections across the $e_i^\perp$, where $\{e_i\}$ is the standard basis.

Induct on n. $n=1$ is obvious.
Let $A \in O(p,q)$, $v \in V$ s.t. $\langle v,v\rangle \neq 0$

- can assume $\langle Av-v, Av-v\rangle \neq 0$
  - otherwise, replace A with $A\cdot(-I) = -A$

Let $x = Av - v$, ρ_x the reflection across $x^\perp$

Since $v = \underbrace{\tfrac{1}{2}(Av+v)}_{(*) \Rightarrow \text{ this is } proj(v)} - \tfrac{1}{2}(Av-v)$ and $\overbrace{\langle Av+v, Av-v\rangle = 0}^{\text{so they are perp. } (*)}$
$= \langle Av, Av\rangle - \langle v,v\rangle = 0$

$\rho_x(v) = 2\,proj(v) - v = Av + v - v = Av.$

$\rho_x(v) = Av \Rightarrow \rho_x(Av) = v$, so $(\rho_x \circ A)(v) = v$
$\rho_x(\rho_x(v)) = v$

$\Rightarrow (\rho_x \circ A)\big|_{v^\perp} \in O(v^\perp, \langle\cdot,\cdot\rangle_{(p,q)}\big|_{v^\perp})$
$= O(v^\perp, \langle\cdot|\cdot\rangle_{(p',q')})$
for some $p'+q' = n-1$.

such p', q' exist for any non-degenerate bilinear form. I.e., $\exists$ basis $\{e_i\}$ s.t.
$\langle e_i, e_j\rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \leq p' \\ -1 & i = j > p' \end{cases}$

by induction $(\rho_x \circ A)\big|_{v^\perp}$ is generated by reflections in $O(v^\perp, \langle\cdot,\cdot\rangle)$, and all such reflections extend to V.
$\therefore O(p,q)$ is generated by reflections. $\square$

Let $O(M^n) \leq O(n,1)$ be the subgroup consisting of maps that keep M^n invariant,

and $SO(M^n) = O(M^n) \cap SL(n+1, \mathbb{R})$.

These are closed subgroups of $GL(n+1, \mathbb{R})$, hence Lie groups.

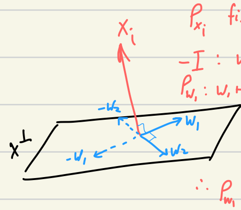
Prop: $O(M^n)$ is generated by the reflections it contains.

If $\langle x, x\rangle \neq 0$, then ρ_x keeps $M^n \cup (-M^n)$ invariant (since for these $\langle v,v\rangle = -1$), and ρ_x

exchanges M^n and $-M^n$ $\iff$ $\langle x, x\rangle < 0$.

In the case $\langle x,x\rangle < 0$, we can replace ρ_x with $-B$, where B is a product of reflections fixing M^n.

(see B+P for details).



ρ_{x_i} fixes w_1, w_2

$-I : w_i \mapsto -w_i$

$\rho_{w_1} : w_1 \mapsto -w_1, \; w_2 \mapsto w_2$

$\rho_{w_2} : w_2 \mapsto -w_2$
$w_1 \mapsto w_1$

$\therefore \rho_{w_1} \circ \rho_{w_2} \circ (-I)$
fixes w_1, w_2,
sends $x_i \mapsto -x_i$

Thm: $\mathrm{Isom}(\mathbb{M}^n) \cong O(M^n)$ and $\mathrm{Isom}^+(\mathbb{M}^n) = SO(M^n)$.
In particular, $\mathrm{Isom}(\mathbb{M}^n)$ is generated by reflections.

proof: Clearly, for any $A \in O(M^n)$, $A|_{\mathbb{M}^n}$ is an isometry, since $A \in O(n,1)$ — linear maps are their own derivatives

Now, let $f \in \mathrm{Isom}(\mathbb{M}^n)$, $x \in \mathbb{M}^n$, and define

$$A : \mathbb{R}^{n+1} \cong \mathbb{R}\langle x \rangle \oplus \mathbb{R}^n \langle x^\perp \rangle \longrightarrow \mathbb{R}^{n+1}$$
$$\lambda x + v \longmapsto \lambda f(x) + df_x(v)$$

$A \in O(M^n)$: $\langle \lambda f(x) + df_x(v), \lambda f(x) + df_x(w) \rangle$

$$= \lambda^2 \langle f(x), f(x) \rangle + \underbrace{\langle \lambda f(x), df_x(w) + df_x(v) \rangle}_{=0 \text{ since } df_x(v) \in f(x)^\perp,\ df_x(w) \in f(x)^\perp}$$
$$+ \langle df_x(v), df_x(w) \rangle$$

$$= \lambda^2 \langle x, x \rangle + \underbrace{\langle \lambda x, v + w \rangle}_{=0} + \langle v, w \rangle$$

$$= \langle \lambda x + v, \lambda x + w \rangle \Rightarrow A \in O(n,1) \Rightarrow A \in O(M^n)$$

Since $f(x) = Ax$ and $df_x = A|_{T_x\mathbb{M}^n}$, f must be $A|_{\mathbb{M}^n}$ (isometries are determined locally)

if A and B are equal on a basis, then they are equal on $\mathbb{R}^{n+1}$!

$\therefore \mathrm{Isom}(\mathbb{M}^n) = \{ A|_{\mathbb{M}^n} : A \in O(M^n) \}$

Since $\mathrm{span}\{M^n\} = \mathbb{R}^{n+1}$, the map $A \longmapsto A|_{M^n} \in \mathrm{Isom}(\mathbb{M}^n)$ is one-to-one, so a group isomorphism.

Isometries of $\mathbb{D}^n$ and $\mathbb{U}^n$:

recall: $f : M \to N$ is conformal if

$$\langle df_x(v),\ df_x(w) \rangle_{f(x)} = a(x) \langle v, w \rangle_x$$

for some differentiable positive fn. $a(x)$, all $x \in M$, $v, w \in T_x M$.

(i.e., f preserves angles, not nec. lengths).

motivate by stating Thm on next page

Lemma: With the Euclidean inner product on D^n, $\pi : \mathbb{M}^n \to D^n$ is conformal.

proof: $\pi : \mathbb{M}^n \to D^n$ is defined by

$$\pi(x,t) = \frac{x}{1+t}, \quad x \in \mathbb{R}^n,\ (x,t) \in \mathbb{R}^{n+1}$$

so $d\pi_{(x,t)}(y,s) = \dfrac{y}{1+t} - \dfrac{sx}{(1+t)^2}$, and

$$\left\langle d\pi_{(x,t)}(y,s),\ d\pi_{(x,t)}(z,r) \right\rangle = \left\langle \frac{y}{1+t} - \frac{sx}{(1+t)^2} \,\middle|\, \frac{z}{1+t} - \frac{rx}{(1+t)^2} \right\rangle$$

$$= \frac{\langle y|z\rangle}{(1+t)^2} - \frac{r\langle y|x\rangle}{(1+t)^3} - \frac{s\langle x|z\rangle}{(1+t)^3} + \frac{sr\langle x|x\rangle}{(1+t)^4}$$

$$= \frac{\langle y|z\rangle}{(1+t)^2} - \frac{rst}{(1+t)^3} - \frac{rst}{(1+t)^3} + \frac{rs(t^2-1)}{(1+t)^4} = \frac{1}{(1+t^2)}\left(\langle y|z\rangle - rs\right)$$

$$= \frac{\langle (y,s)|(z,r)\rangle}{(1+t)^2} \qquad \square$$

$\boxed{x_1^2 + \dots + x_n^2 - t^2 = -1}$

similarly ...

since $\langle x|x\rangle = t^2 - 1$, $\langle x, y\rangle = st$, $\langle x, z\rangle = rt$

Theorem: $\mathrm{Isom}(\mathbb{D}^n) \cong \mathrm{Conf}(D^n)$

$\mathrm{Isom}(\mathbb{U}^n) \cong \mathrm{Conf}(U^n)$

conformal map → w.r.t. Euclidean metric.

proof:

($\subseteq$) since isometries are conformal,

$\mathrm{Isom}(\mathbb{D}^n) \subseteq \mathrm{Conf}(D^n)$ by the Lemma.

↳ $\pi \circ f \circ \pi^{-1}$, $f \in \mathrm{Isom}(\mathbb{M}^n)$

Thus since sphere inversions are conformal,

$\mathrm{Isom}(\mathbb{U}^n) \subseteq \mathrm{Conf}(U^n)$.

($\supseteq$): We will use:

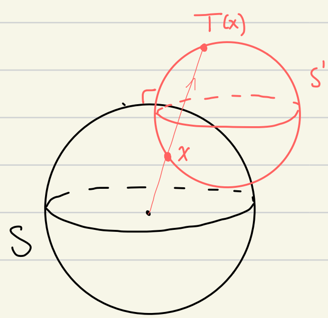
Fact: $\mathrm{Conf}(D^n)$ is generated by sphere inversions fixing $\partial D^n = S^{n-1}$, and $\mathrm{Conf}(U^n)$ is generated by sphere inversions fixing $\partial U^n = \mathbb{R}^{n-1} \cup \{\infty\}$

(here a reflection is considered a sphere inversion in the sphere $\{\text{plane}\} \cup \{\infty\}$)

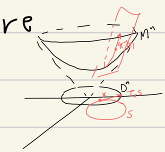
$\therefore$ suffices to show sphere inversions are isometries

Recall: Inversion in a sphere S:

(given x, let S' be a sphere s.t. $x \in S'$ and $S \perp S'$. Then $T(x)$ is in S', and on the line thru x and center of S.)



Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a sphere inversion fixing D^n (setwise). Then T is inversion in a sphere $S = S^{n-1}$ orthogonal to ∂D^n, and this sphere is fixed pointwise by T. Let $x \in S \cap D^n$. Then $\pi^{-1}(x)$, $d\pi^{-1}_x(T_x S)$ determine an n-plane in $\mathbb{R}^{n+1}$ through the origin. Let A be reflection in this plane. Then $\pi \circ A \circ \pi^{-1}$ is conformal, hence gen. by inversions, and fixes S (since it fixes x and $T_x S$). $\therefore$ $\pi \circ A \circ \pi^{-1} = T$, so $T \in \mathrm{Isom}(\mathbb{D}^n)$.



needs more → see addendum next page.

Since $i: D^n \to U^n$ is a sphere inversion,

$$T' \in \mathrm{Conf}(U^n) \Rightarrow i^{-1} \circ T' \circ i \in \mathrm{Conf}(D^n)$$

hence $(i^{-1} \circ T' \circ i)$ is an isometry of $\mathbb{D}^n$

$$\therefore T' \in \mathrm{Isom}(\mathcal{U}^n).$$

<u>Corollary</u>: $\mathrm{Isom}^+(\mathbb{D}^2) \cong \mathrm{Isom}^+(\mathcal{U}^2) \cong PSL_2(\mathbb{R})$

$$\mathrm{Isom}^+(\mathbb{D}^3) \cong \mathrm{Isom}^+(\mathcal{U}^3) \cong PSL_2(\mathbb{C})$$

proof:

$\mathrm{Conf}^+(\mathcal{U}^2)$ consists of Möbius transformations $\mathcal{U}^2 \to \mathcal{U}^2$ that fix $\mathbb{R} \cup \{\infty\}$. I.e., maps of the form

$$T: \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : ad-bc \neq 0 \right\} \cong GL_2(\mathbb{C})$$

Addendum to proof of previous Thm:

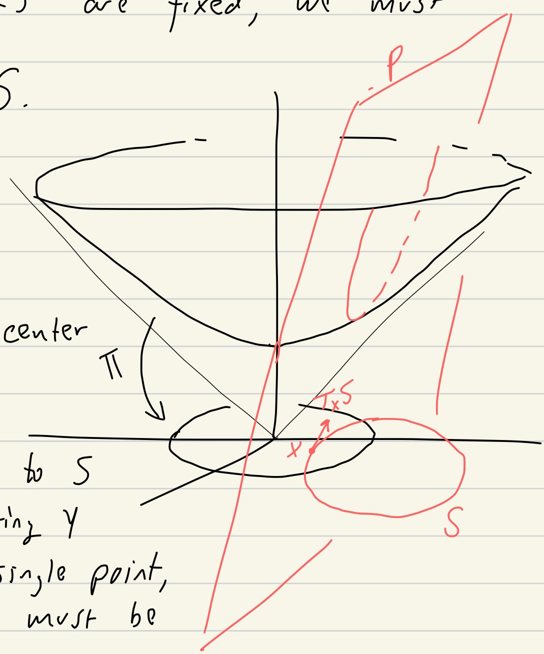
$f = \pi \circ A \circ \pi^{-1}$ fixes x and T_xS.

$\therefore$ it fixes (setwise, for now) S, since S is the unique sphere $\perp$ to ∂D^n and tangent to T_xS at x.

$\therefore$ the two components of $D^n \setminus S$ are exchanged by f since f maps $T_xS^{\perp} \ni v \longmapsto -v$

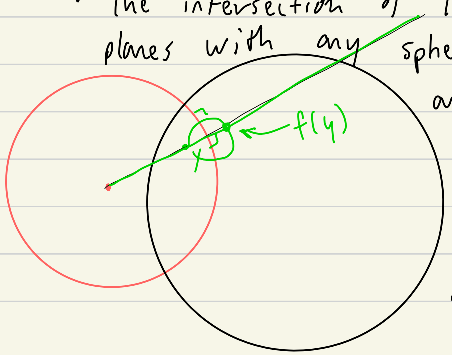
- Since A fixes $P \cap M^n$, $\pi \circ A \circ \pi^{-1}$ fixes $\pi(P \cap M^n)$. Since no points of $D^n \setminus S$ are fixed, we must have

$$\pi(P \cap M^n) = D^n \cap S.$$



- for $y \in D^n \setminus S$

- take $n-1$ $(n-1)$-planes, each passing thru y and center of S, and $\perp$ to S.
- the intersection of these planes with any sphere $\perp$ to S and meeting y is a single point, which must be $f(y)$.



$\therefore$ f is inversion in S.

Note that if $T = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then $T(z) = \frac{\lambda z}{\lambda} = z$,
so T is trivial as a Möbius trans.
Easy to see all others act non-trivially

$$\therefore \{\text{Möbius trans.}\} \cong GL_2(\mathbb{C}) / \lambda I \cong PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})$$

If $T(z) \in \mathbb{R} \cup \{\infty\}$ for all $z \in \mathbb{R} \cup \{\infty\}$, then we must have $a, b, c, d \in \mathbb{R}$

$$\therefore \mathrm{Isom}^+(\mathbb{D}^2) \cong \mathrm{Isom}^+(\mathcal{U}^2) \cong \mathrm{Conf}^+(\mathcal{U}^2) \cong PSL_2(\mathbb{R})$$

<u>Rmk:</u> $PGL_2(\mathbb{R}) \not\cong PSL_2(\mathbb{R})$, since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in PGL_2(\mathbb{R}) \setminus PSL_2(\mathbb{R}).$$

(its the only non-trivial element in $PGL_2\mathbb{R} / PSL_2\mathbb{R}$)

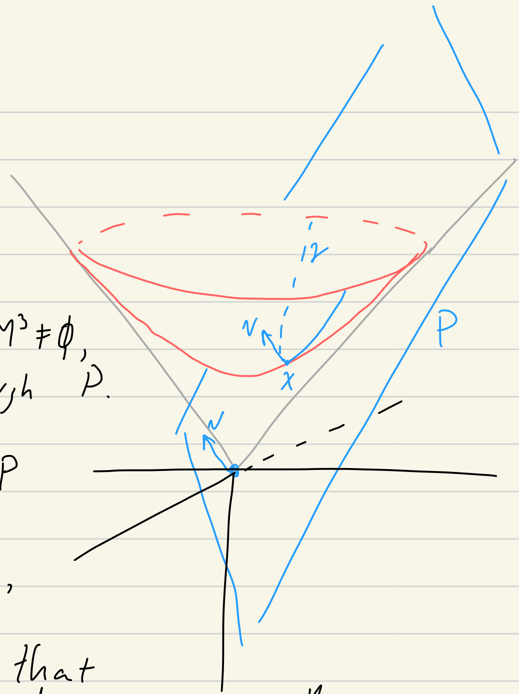
But $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ doesn't fix $\mathcal{U}^2$, so we don't include it.

- $\mathrm{Conf}^+(\mathcal{U}^3) = \mathrm{Conf}^+(\partial \mathcal{U}^3)$ consist of <u>all</u> Möbius transformations, so

$$\mathrm{Isom}^+(\mathbb{D}^3) \cong \mathrm{Isom}^+(\mathcal{U}^3) \cong \mathrm{Conf}(\mathcal{U}^3) \cong PSL_2(\mathbb{C}).$$

$\square$

Geodesics



Let P be a plane in $\mathbb{R}^3$ through the origin, with $P \cap \mathbb{M}^3 \neq \emptyset$, and let A be reflection through P.

Let $x \in \mathbb{M}^2 \cap P$, $v \in T_x \mathbb{M}^2 \cap P$

Then $Ax = x$ and $Av = v$,

So if γ is the geodesic that passes through x in the direction v, then

$A\gamma = \gamma$ (geodesics are uniquely determined locally)

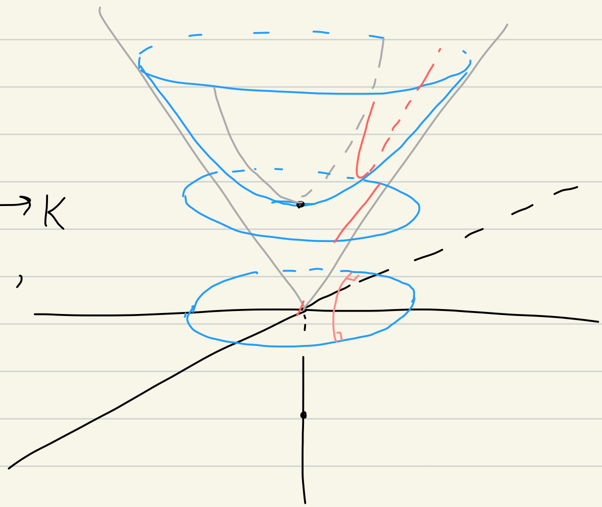
$\therefore \quad \gamma = \mathbb{M}^2 \cap P$

For any $x \in \mathbb{M}^2$ and $v \in T_x \mathbb{M}^2$, there is a unique plane containing x and v. So <u>all</u> geodesics come from intersecting planes with $\mathbb{M}^2$.

Similarly for $\mathbb{M}^n$: geodesics are intersections of 2-planes with $\mathbb{M}^n$.

Geodesics in other models:

- Klein model:

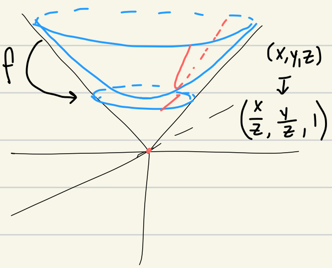
  - Since the map $f : \mathbb{M}^2 \to K$ is just scaling by $1/x_3$, geodesics are just

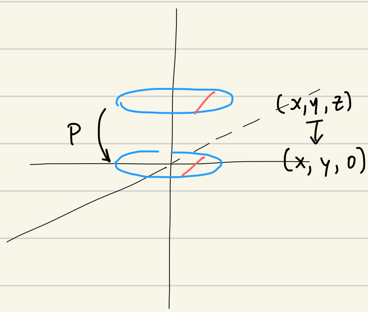
    $$K \cap P \quad \text{for planes}$$

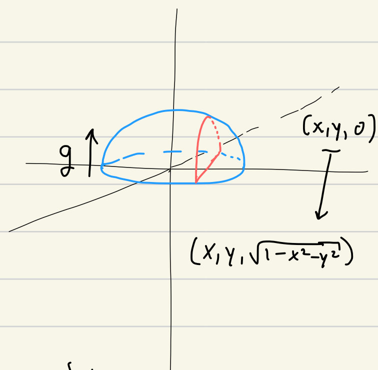
    P through the origin.

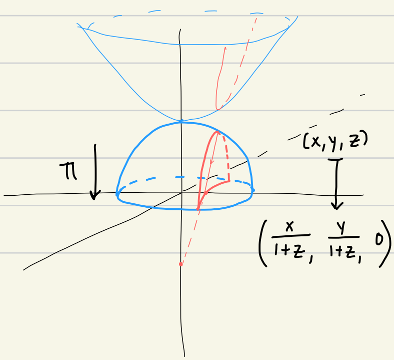
    $\therefore$ geodesic are straight lines.

- Poincaré disk model:









Claim: $\left(\pi \circ g \circ p \circ f\right)\big|_{\mathbb{M}^2} = \pi\big|_{\mathbb{M}^2}$

proof: easy computation.

- Claim $\Rightarrow$ geodesics on $\mathbb{D}^2$ are arcs of circles perpendicular to $\partial\mathbb{D}^2$.

  Why?: g maps geodesics of $\mathbb{K}^2$ to semicircles on S^2. Since stereographic projection maps circles on S^2 to circles in $\mathbb{R}^2$, $\pi \circ g(\gamma)$ must be an arc of a circle, for γ a geodesic of $\mathbb{K}^2$.

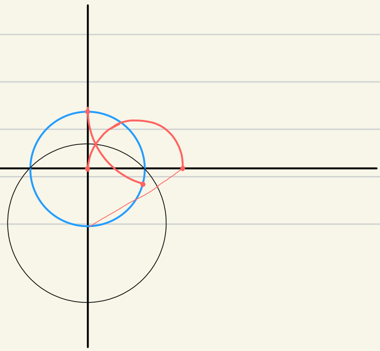
  Since $g(\gamma)$ is clearly orthogonal to $\partial\mathbb{D}^2$, and π is conformal and fixes $\partial\mathbb{D}^2$, $\pi \circ g(\gamma)$ must also be $\perp$ to $\partial\mathbb{D}^2$.

- Upper half-space model:

  $i: \mathbb{D}^2 \rightarrow \mathbb{U}^2$ is a circle inversion, so it is conformal and maps circles/lines to circles/lines

  $\therefore$ geodesics of $\mathbb{H}^2$ are semi-circles perpendicular to the x-axis, and vertical lines.

(Note: same argument works for $\mathbb{D}^n$, $\mathbb{U}^n$, $\mathbb{K}^n$, and for totally geodesic subspaces of same.)



2.1: Hyperbolic 2-space $\mathbb{H}^2$

Notation: from here on, $\mathbb{H}^2$ will denote hyperbolic 2-space, regardless of the model used.

Going forward we will most often use the upper half-space model, and use complex coordinates:

$$\mathbb{H}^2 = \{ x + iy \in \mathbb{C} : y > 0 \}$$

Def-n: The boundary (at infinity) of $\mathbb{H}^2$ is $\mathbb{R} \cup \{\infty\}$ for the upper half-plane model, or ∂D^2 for Poincaré disk and Klein models. Denoted by S^1_∞, $\partial_\infty \mathbb{H}^2$, or $\partial \mathbb{H}^2$. $(\partial \mathbb{H}^2 \not\subset \mathbb{H}^2)$

One can check that the metric in the upper half-plane model is given by $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ (exercise)

More precisely: given $(x,y) \in \mathbb{H}^2$, and $v \in T_{(x,y)}\mathbb{H}^2$, we can write $v = v_x \dfrac{\partial}{\partial x} + v_y \dfrac{\partial}{\partial y}$, or as a vector

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}.$$

Then the metric is given by

$$\langle v, w \rangle = (v_x, v_y) \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix}.$$

1-29-20

- **Arc length:** Let $\gamma(t)$ be a differentiable curve in $\mathbb{H}^2$, with $t \in [a,b]$.

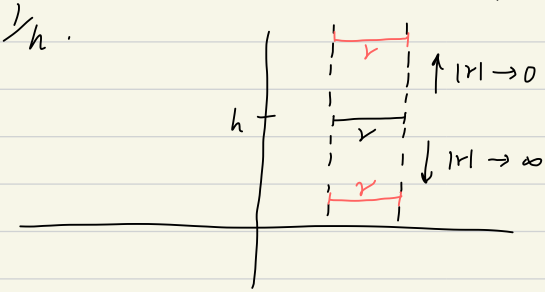
then
$$|\gamma| = \int_a^b \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}\, ds$$

Writing $\gamma(t) = (\gamma_x(t), \gamma_y(t))$, we then get
$$|\gamma| = \int_a^b \sqrt{(\gamma_x'(s))^2 + (\gamma_y'(s))^2} \cdot \frac{1}{\gamma_y(s)}\, ds \qquad (*)$$

Ex: Fix $h > 0$, and define $\gamma(t) = (t, h)$, $t \in [0,1]$.

$(*)$ gives $|\gamma| = \int_0^1 1 \cdot \frac{1}{h}\, ds = \frac{1}{h}$

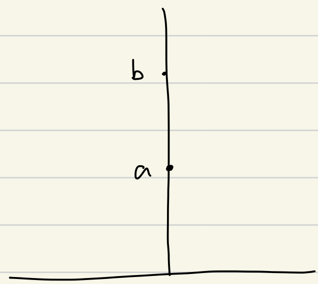
i.e., the length of γ is its Euclidean length, scaled by $1/h$.



Ex: Let γ be a vertical line from (x,a) to (x,b), for x fixed.

$$\therefore \quad \gamma(t) = (x,t), \quad t \in [a,b]$$

$$\gamma'(t) = (0,1)$$



$$|\gamma| = \int_a^b 1 \cdot \frac{1}{s}\, ds = \log\left(\frac{b}{a}\right)$$

note: as $a \to 0$, $|\gamma| \to \infty$
as $b \to \infty$, $|\gamma| \to \infty$

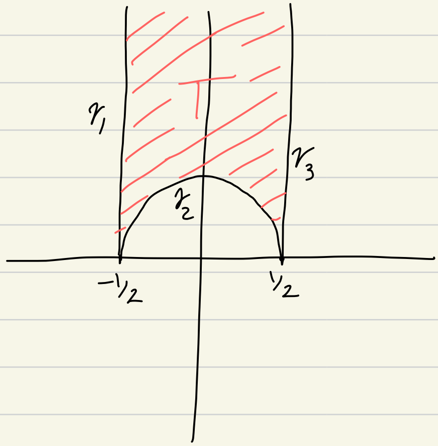
Area:

In general, for a Riemannian mfld. M, if $R \subseteq M$ is contained in a chart with coordinates $(x_1, \ldots, x_n)$ and metric g_{ij}, then

$$vol(R) = \int_R dvol = \int_R \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n$$

Thus for $\mathbb{H}^2$ we have

$$area(R) = \int_R \frac{1}{y^2}\, dx\, dy$$



Ex: Ideal triangle:

Let T be the ideal triangle bounded by the geodesics γ_1, γ_2, γ_3 shown.

$$\text{area}(T) = \int_T \frac{1}{y^2}\,dx\,dy$$

$$= 2\int_{-1/2}^{0}\int_{\sqrt{\frac{1}{4}-x^2}}^{\infty} \frac{1}{y^2}\,dy\,dx = 2\int_{-1/2}^{0} \frac{-1}{\sqrt{\frac{1}{4}-x^2}}\,dx$$

$$= 2\cdot\arcsin(2x)\Big|_{-1/2}^{0} = 2(-\arcsin(-1)) = \pi.$$

- Infinite Euclidean area, but finite hyperbolic area!

- soon: every ideal triangle has area π (all are isometric).

<u>Lemma</u> 2.7: Given any three distinct points $z_1, z_2, z_3 \in \partial \mathbb{H}^2$, there exists $T \in \mathrm{Isom}^+(\mathbb{H}^2)$

s.t. $Tz_3 = \infty$, and $\{Tz_1, Tz_2\} = \{0, 1\}$

proof: If necessary, switch z_1 and z_2 so that z_1, z_2, z_3 are arranged clockwise around $\partial \mathbb{H}^2$.

If no $z_i = \infty$:

$$T = \begin{pmatrix} z_1 - z_3 & -z_2(z_1 - z_3) \\ z_1 - z_2 & -z_3(z_1 - z_2) \end{pmatrix}$$

$$T : z \longmapsto \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

maps $z_1 \mapsto 1$, $z_2 \mapsto 0$, $z_3 \mapsto \infty$.

$$\det(T) = (z_1 - z_3)(z_1 - z_2)(z_2 - z_3) > 0$$

since points are arranged clockwise.

If $z_1 = \infty$, $z_2 = \infty$, or $z_3 = \infty$, then

$$z \mapsto \frac{z - z_2}{z - z_3} \; ; \quad z \longmapsto \frac{z_1 - z_3}{z - z_3} \; ; \quad z \mapsto \frac{z - z_2}{z_1 - z_2}$$

respectively, are the desired isometries.

$\square$

Corollary: All idea triangles are isometric, and have area π

proof: take vertices to $0, 1, \infty$ by an isometry, then to $-1/2, 1/2, \infty$.

Lemma 2.8: Two distinct geodesics γ_1 and γ_2 in $\mathbb{H}^2$ either

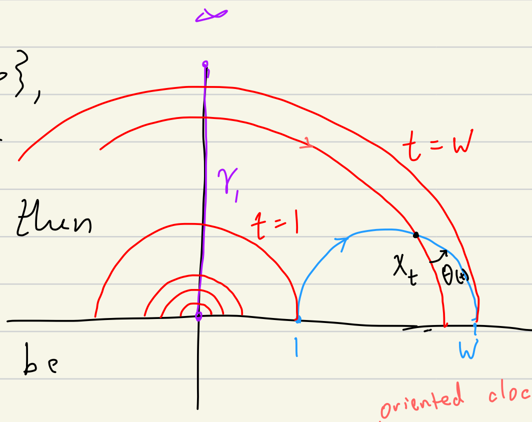
(1) Intersect in a single point in $\mathbb{H}^2$
(2) Intersect in a single point of $\partial\mathbb{H}^2$, or
(3) are disjoint in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$, and there exists a geodesic γ_3 $\perp$ to both γ_1 and γ_2

note: Purcell gives a more constructive proof

proof: Using the lemma, we may assume that $\gamma_1(0) = 0$ and $\gamma_1(1) = \infty$ and $\gamma_2(0) = 1$.

If $\gamma_2(1) \in \{0, \infty\}$, then (2) holds.

If $\gamma_2(1) \in \mathbb{R}_{<0}$, then (1) holds



Otherwise, let g_t be the geodesic that is the top half of the circle centered at 0 with radius t. Note that $g_t \perp \gamma_1$ $\forall t$.

oriented clockwise

at $t=1$ and $t=w$, extend g_t and γ_2 as necessary to define $\theta(t)$

For $t \in (1, w)$, let $g_t \cap \gamma_2 = x_t$. Let $\theta(t)$ be the angle between two vectors at x_t tangent to g_t and γ_2 and oriented clockwise along each.
$\theta(w) = 0$ and $\theta(1) = \pi$, and $\theta(t)$ is clearly continuous. $\therefore$ $\theta(t) = \pi/2$ for some $t \in (1, w)$. $\square$

2.3: hyperbolic 3-space $\mathbb{H}^3$

We will most often use the upper half-space model:

$$\mathbb{H}^3 = \{(x+iy, t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}; \quad \partial\mathbb{H}^3 := \mathbb{C} \cup \{\infty\}$$

One can calculate the metric as the pull-back of the Minkowski metric via $\pi^{-1} \circ i^{-1}$ to be:

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$$

We have shown:

Thm 2.14: geodesics in $\mathbb{H}^3$ consist of vertical lines and semi-circles orthogonal to $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. Totally geodesic planes are vertical planes and hemispheres centered on $\mathbb{C}$.

Thm 2.15. $\mathrm{Isom}(\mathbb{H}^3)$ is generated by inversions in spheres orthogonal to $\mathbb{C}$, and $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$ acts on $\partial\mathbb{H}^3$ as Möbius transformations.

Lemma 2.15a: If $f \in \mathrm{Isom}^+(\mathbb{H}^3)$ fixes 3 points in $\partial \mathbb{H}^3$, then $f = 1$.

proof: $$\frac{az+b}{cz+d} = z \implies cz^2 + (d-a)z - b = 0$$

If $c=b=0$ and $a=d=1$, then this holds for all z, so $f=1$.

If $f(\infty)=\infty$, then $c=0$, so $z = \frac{b}{d-a}$ has at most one other solution (may be ∞ if $d-a=0$, $b\neq 0$).

If $c \neq 0$ then $f(\infty) \neq \infty$, and $cz^2 + (d-a)z - b = 0$ has at most 2 solutions in $\mathbb{C}$.

Corollary: If $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ fixes 3 points in $\mathbb{H}^2 \cup \partial \mathbb{H}^2$, then $f = 1$.

Lemma 2.15b: Given any triple of points $z_1, z_2, z_3 \in \partial \mathbb{H}^3$, there exists unique $f \in \mathrm{Isom}^+(\mathbb{H}^3)$ s.t.

$$f(z_1) = 0, \quad f(z_2) = 1, \quad f(z_3) = \infty$$

proof: Existence: similar to Lemma 2.7.

Uniqueness: If g sends $z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$, then $f^{-1} \circ g$ fixes z_1, z_2, z_3 by Lemma 2.15a

$$\begin{array}{c} SL_2\mathbb{C} \\ \big\downarrow {\scriptstyle /\pm I} \\ PSL_2\mathbb{C} \\ \text{\rotatebox{90}{$\cong$}} \\ \mathrm{Isom}^+(\mathbb{H}^3) \end{array}$$

Thm 2.16: Let $f \in \mathrm{Isom}^+(\mathbb{H}^3)$, regarded as an element of $SL_2(\mathbb{C})$. If $f \neq I$ in $PSL_2\mathbb{C}$, then one of the following holds

1) Parabolic:
   - f has exactly one fixed point in $\partial\mathbb{H}^3$ (none in $\mathbb{H}^3$)
   - $\mathrm{Tr}(f) = \pm 2$ — trace of f.
   - f is conjugate to $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in $PSL_2\mathbb{C}$, for some $x \in \mathbb{C}$.

2) Elliptic:
   - f has exactly two fixed points on $\partial\mathbb{H}^3$, and fixes pointwise the geodesic axis between them
   - $\mathrm{Tr}(f) \in \mathbb{R}$, $|\mathrm{Tr}(f)| < 2$
   - f is conjugate to $\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$ in $PSL_2\mathbb{C}$, $\theta \in \mathbb{R}$.

3) Loxodromic:
   - f has exactly two fixed points on $\partial\mathbb{H}^3$, and fixes (setwise) the geodesic axis between them.
   - $\mathrm{Tr}(f) \in \mathbb{C}$ or $\underbrace{\mathrm{Tr}(f) \in \mathbb{R} \text{ and } |\mathrm{Tr}(f)| > 2}_{\text{hyperbolic (no rotation).}}$
   - f is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ in $PSL_2\mathbb{C}$, for some $\lambda \in \mathbb{C}$.

<u>proof</u>: By Lemma 2.15a, $f \neq 1 \Rightarrow f$ fixes 0, 1, or 2 points on $\partial \mathbb{H}^3$.

As before, $\dfrac{az+b}{cz+d} = z \Rightarrow cz^2 + (d-a)z - b = 0$

If this equation has no solutions, then $c=0$ and $d-a=0$. In this case $f \neq 1 \Rightarrow b \neq 0$, so $f(\infty) = \infty$.
$\therefore f$ fixes exactly 1 or 2 points on $\partial \mathbb{H}^3$.

Case 1: f fixes 1 point $p \in \partial \mathbb{H}^3$.

- by Lemma 2.15b, $\exists\, g$ s.t. $g \circ f \circ g^{-1}(\infty) = \infty$, where $g(p) = \infty$. Let

$$g \circ f \circ g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

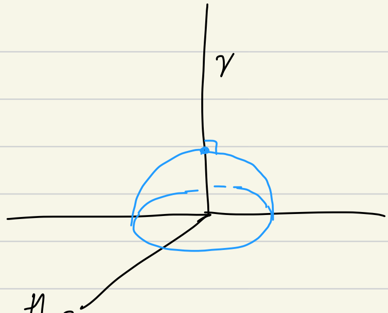
Then we have $d - a = 0$ (o/w f fixes $b/_{d-a} \neq \infty$).

$$\therefore g \circ f \circ g^{-1} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \text{ in } PSL_2\mathbb{C}$$

$$\therefore Tr(f) = Tr(g \circ f \circ g^{-1}) = \pm 2$$

Case 2: f fixes 2 points $p, q \in \partial \mathbb{H}^3$, and the geodesic γ from p to q is fixed pt-wise.

- again by Lemma 2.15b, may assume after conjugating that $p = 0$, $q = \infty$.



Since f fixes ∞, $c = 0$.
Since f fixes 0, $b = 0$ also

$\therefore \ f : z \longmapsto \dfrac{az}{d}$

The hemisphere H formed by the sphere centered at 0 of radius 1 is orthogonal to γ, and $x = H \cap \gamma$ is fixed by f.

$\therefore \ f(H) = H$ since H is totally geodesic, and no other totally geodesic subspace is $\perp$ to γ at x.

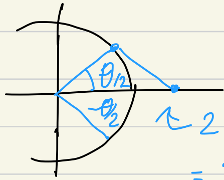
$\therefore \ f$ fixes the unit circle in $\mathbb{C}$.

$\Rightarrow \left|\dfrac{a}{d}\right| = 1$, so $\dfrac{a}{d} = e^{i\theta}$ for some $\theta \in \mathbb{R}$

$f \in SL_2\mathbb{C} \Rightarrow a \cdot d = 1$, so $a = \dfrac{1}{d}$

$$\therefore \ f = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \text{ up to } \pm I.$$

$$|\mathrm{Tr}(f)| = \left|e^{i\theta/2} + e^{-i\theta/2}\right| = |\underbrace{2\cos(\theta/2)}_{\in \mathbb{R}}| < 2$$



$\leftarrow 2\,\mathrm{Re}(e^{i\theta/2})$
$= 2\cos(\theta/2)$
(or use Euler's formula)

Case 3: f fixes 2 points $p, q \in \partial \mathbb{H}^3$, and fixes setwise (but <u>not</u> pt.-wise), the axis from p to q.

Again, may take $p = 0$, $q = \infty$

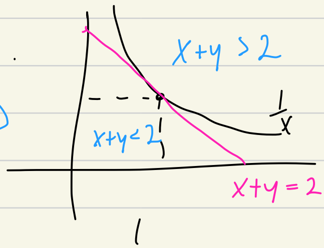
As before, $c = b = 0$, so $f: z \longmapsto \frac{az}{d}$.

$f \in SL_2\mathbb{C} \Rightarrow \det(f) = a \cdot d = 1 \Rightarrow d = \frac{1}{a}$, so

$$f = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \quad a \in \mathbb{C}.$$

If $\operatorname{Tr}(f) \in \mathbb{R}$, then $a \in \mathbb{R}$, and

$$\underline{a + \frac{1}{a} > 2} \quad (\text{for all } a \in \mathbb{R}).$$



Otherwise, $\operatorname{Tr}(f) \in \mathbb{C}$.

<u>Remark</u>: If $a \in \mathbb{R}$, the $f = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ acts as a dilation centered at 0.

Otherwise, $a = re^{i\theta/2}$, so

$$f = \begin{pmatrix} re^{i\theta/2} & 0 \\ 0 & \frac{1}{r}e^{-i\theta/2} \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \cdot \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix},$$

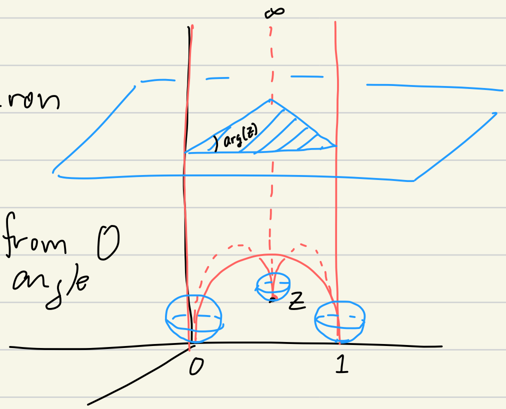
so f acts as a rotation, followed by a dilation. I.e., a screw motion along γ.

Def-n: An ideal tetrahedon in $\mathbb{H}^3$ is a tetrahedron with vertices on $\partial\mathbb{H}^3$. i.e., the convex hull of 4 points of $\partial\mathbb{H}^3$.

Since $\mathrm{Isom}^+(\mathbb{H}^3)$ acts triply transitively on $\partial\mathbb{H}^3$, any tetrahedron is isometric to one with vertices at $0, 1, \infty$, and z for some $z \in \mathbb{C}$.

Let T be an ideal tetrahedron with vertices $0, 1, \infty, z$.

If e is the edge of T from 0 to ∞, then the dihedral angle at e is $\arg(z)$.



Definition: A horoball in $\mathbb{H}^3$ is a set

$$\{(z,t) \in \mathbb{H}^3 : t \geq h\}$$

for some $h \in \mathbb{R}_{>0}$, or the image of such a set under an isometry of $\mathbb{H}^3$ (i.e., a ball tangent to $\partial\mathbb{H}^3$).

A horosphere is the boundary of a horoball

The center of a horoball or horosphere is its point of tangency on $\partial\mathbb{H}^3$.

- Since any horosphere is isometric to a horizontal plane, the induced metric on a horosphere is Euclidean.

  $\therefore$ The link of a vertex of a tetrahedron is a Euclidean triangle.

Chapter 3: Geometric Structures on mflds.

· Polyhedral decompositions:

Def-n: A n-dimensional polyhedral gluing X consists of a collection of polyhedra $P_1, \ldots, P_K$, and gluing maps $\{\varphi_i\}_i$ s.t. each φ_i is a homeo-sm between codim-1 faces that maps codim-j faces to codim-j faces, such that

$$P_1 \sqcup \cdots \sqcup P_K / \{\varphi\}_i \cong X$$

Prop-n: A 3-dimensional polyhedral gluing yields a manifold if and only if the link of every material vertex is homeomorphic to S^2, and no edge is glued to its reverse.
non-ideal.

proof: Exercise.

Def-n: If a polyhedral gluing gives a mfld M, then we'll say M has a (topological) polyhedral decomposition.

Def-n: A geometric polyhedral decomposition of a mfld M is a topological polyhedral decomp-n s.t.
(1) Each polyhedron has a metric (i.e., they're embedded in a common metric space)
(2) gluing maps are isometries
(3) the gluing induces a complete, smooth metric on M

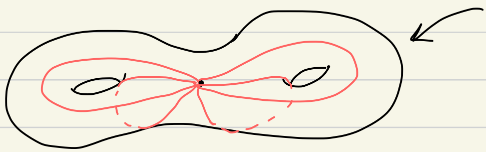
$\boxed{\text{Every Cauchy sequence in } M \text{ converges in } M.}$

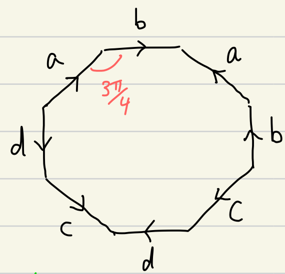
Lemma: If M has a polyhedral decomposition into hyperbolic polyhedra such that gluing maps are isometries, then the gluing induces a smooth metric on M $\iff$ the nbhd. of each point of M (in the quotient topology) is isometric to a ball in $\mathbb{H}^n$.

proof: immediate

- I.e., in 2-dimensions, angles around a vertex sum to 2π.

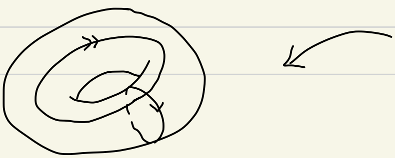
Ex: Genus 2-surface (topological)

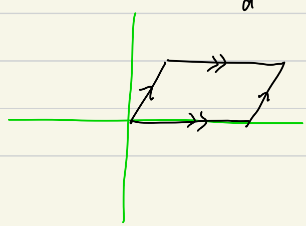




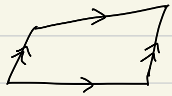
$\Rightarrow$ not a geometric gluing!

Ex: Euclidean Torus:





non-Ex: Affine torus: things go wrong!



• angle sum is OK, but gluing is not by isometries.

· Geometric Structures

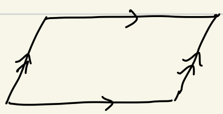
Definition 3.3: Let X be a manifold, and let G be a group (of real analytic diffeomorphisms) acting (transitively) on X. We say that a mfld. M has a (G,X)-structure if $\forall\, x \in M$, $\exists$ a chart (U,φ), $\varphi : \underset{\substack{\Psi \\ x}}{U} \longrightarrow \varphi(U) \subseteq X$, and if two charts (U,φ) and (V,ψ) overlap, then

$$\gamma = \varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$$

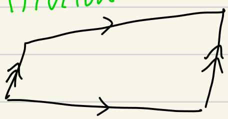
restricts to an element of G on connected components of $\psi(U \cap V)$.

non-Ex: • $(C^\infty, \mathbb{R}^n)$-structure: smooth manifolds.
"smooth structure"

Examples: • $(\mathrm{Isom}(\mathbb{E}^2), \mathbb{E}^2)$-structure:

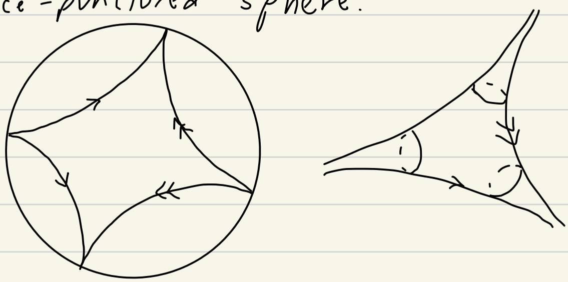


- $(\mathrm{Aff}(\mathbb{R}^2), \mathbb{R}^2)$-structure: $Ax+b$: scaling, shearing, rotation, translation.
  "Affine structure"

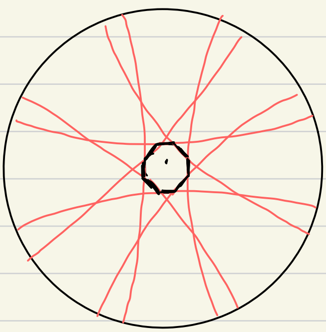


- $(\mathrm{Isom}(\mathbb{H}^2), \mathbb{H}^2)$-structure: "hyperbolic structure"

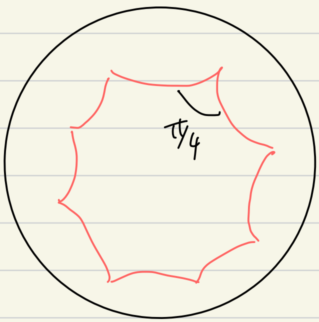
Thrice-punctured sphere:



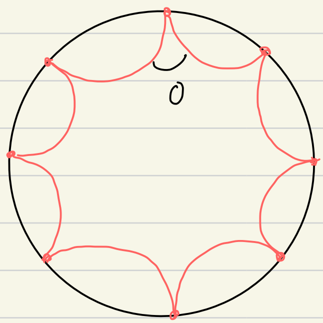
Genus 2 surface:



int. angle $\sim 3\pi/4$
total angle at vertex $> 2\pi$



total angle
$8 \cdot \pi/4 = 2\pi$



total angle at vertex 0

Remark: If M is obtained as a gluing of hyperbolic polyhedra by isometries, and the induced metric on M is smooth, then we get a $(\mathbb{H}^n, \mathrm{Isom}(\mathbb{H}^n))$-structure: isometric balls in M lift via the quotient map to balls in $\mathbb{H}^n$, and transition maps are gluing isometries.

Complete Structures: (3.2)

Q: When does a gluing of hyp-c polyhedra yield a complete metric?

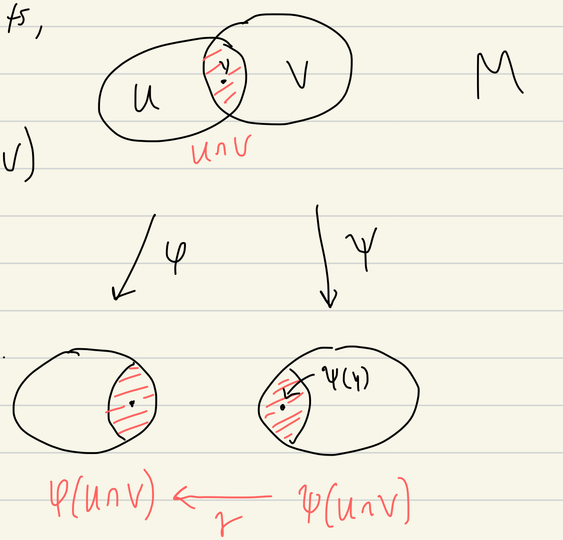
- no ideal vertex $\rightsquigarrow$ always.

otherwise, ...

(3.2.1) Developing Map and Completeness.

Let M be a (G, X)-manifold, and let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be coordinate charts for M.

Let (U, φ), (V, ψ) be charts, $y \in U \cap V$.



$$\gamma = \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

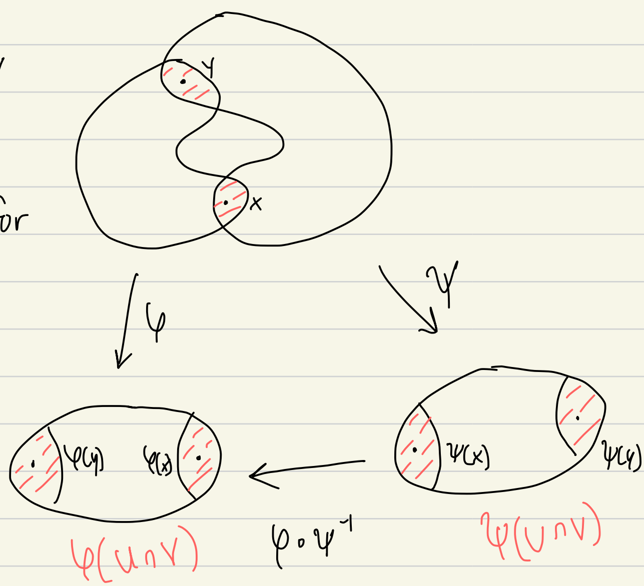
agrees in a nbhd. of y with an element of G, by the def-n of a (G, X)-structure.

Let $\gamma(y)$ be this element of G. Define a map

$\Phi : U \cap V \longrightarrow X$ by

$$\Phi(x) = \begin{cases} \varphi(x) & \text{if } x \in U \\ \gamma(y)\psi(x) & \text{if } x \in V \end{cases}$$

- If $U \cap V$ is connected, then $\varphi(x) = \gamma(y) \cdot \psi(x)$ for all $x \in U \cap V$, so Φ is well defined.



- If $U \cap V$ is not connected, Φ may not be well-defined.

- We could ensure that $U \cap V$ is connected by refining the atlas, but we would still run into problems going forward, as we extend Φ with more charts.

Solution: work in the universal cover.

Let $\alpha : [0,1] \longrightarrow M$ be a path representing a point $[\alpha] \in \tilde{M}$.

Let $0 = t_0 < t_1 < \dots < t_n = 1$, (U_i, φ_i) satisfy

$$\alpha([t_i, t_{i+1}]) \subseteq U_i \quad \text{for } i = 0, 1, \dots, n-1$$

Let $x_i = \alpha(t_i)$, so that x_0 is the basepoint.

Note that $x_i \in U_{i-1} \cap U_i$.

Let $\gamma_{i-1,i} = \varphi_{i-1} \circ \varphi_i^{-1}$. $\gamma_{i-1,i}$ restricts to some $\boxed{\gamma_{i-1,i}(x_i) \in G}$ on the connected component of $\varphi_i(U_{i-1} \cap U_i)$ containing $y_i = \varphi_i(x_i)$

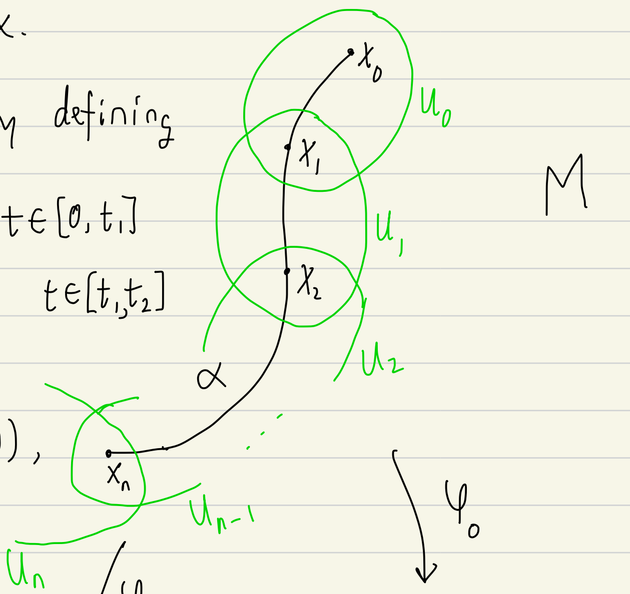
This assignment needs analyticity, though strictly speaking we don't need analyticity here for well-definedness of the map $\gamma_{i-1,i}(x_i)\big|_{U_{i-1} \cap U_i}$.

$\alpha : [0,1] \longrightarrow M$, $\alpha(t_i) = x_i$

We will extend φ_0 along α.

· First step: extend φ_0 by defining
$\Phi_1 : [0, t_2] \longrightarrow X$ by

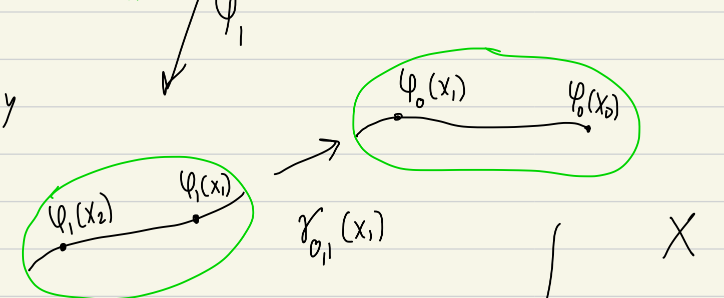
$$\Phi_1(t) = \begin{cases} \varphi_0(\alpha(t)) & t \in [0, t_1] \\ \gamma_{0,1}(x_1)\, \varphi_1(\alpha(t)) & t \in [t_1, t_2] \end{cases}$$



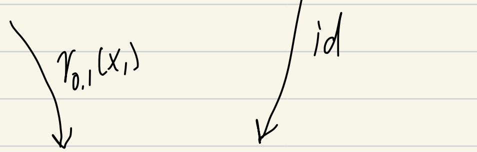
Since $\varphi_0(\alpha(t_1)) = \gamma_{0,1}(x_1)\varphi_1(\alpha(t_1))$,

Φ_1 is well-defined.

Extend inductively to
$\Phi_i : [0, t_{i+1}] \longrightarrow X$ by
setting

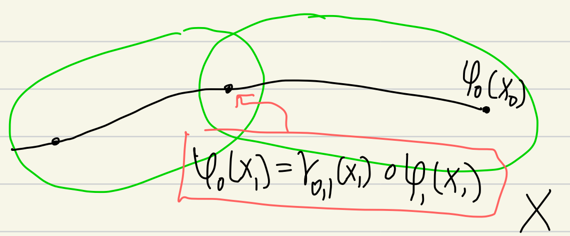


$$\Phi_i(t) = \begin{cases} \Phi_{i-1}(t) & t \in [0, t_i] \\ \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\cdots\gamma_{i-1,i}(x_i)\varphi_i(\alpha(t)) & t \in [t_i, t_{i+1}] \end{cases}$$



Since $\varphi_{i-1}(\alpha(t_i)) = \gamma_{i-1,i}(x_i)\,\varphi_i(\alpha(t_i))$,

we have by induction that



$\Phi_{i-1}(t_i) = \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\cdots\gamma_{i-2,i-1}(x_{i-1})\varphi_{i-1}(\alpha(t_i))$

$= \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\cdots\gamma_{i-2,i-1}(x_{i-1})\gamma_{i-1,i}(x_i)\varphi_i(\alpha(t_i))$, so $\Phi_i(t)$ is well-defined.

At the $(n-1)$th step, we get a map

$$\Phi_{n-1} : [0,1] \longrightarrow X$$

In fact, for some nbhd U of $\alpha(1)$, the composition defining Φ_{n-1} gives a map

$$\Phi_{[\alpha]} : U \to X$$

$$\Phi_{[\alpha]}(x) = \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\cdots\gamma_{n-2,n-1}(x_{n-1})\varphi_{n-1}(x)$$

<u>Exercise</u>: show that the definition of $\Phi_{[\alpha]}$ does not depend on

- choice of α rep-ng $[\alpha]$
- choice of t_i, $i \geq 1$
- choice of charts (U_i, φ_i) for $i \geq 1$

<u>Def-n</u>: The <u>developing map</u> $D : \tilde{M} \longrightarrow X$ is the map

$$D([\alpha]) = \Phi_{[\alpha]}(\alpha(1)) = \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)\cdots\gamma_{n-2,n-1}(x_{n-1})\varphi_{n-1}(\alpha(1))$$

- Note: in a nbhd of $[\alpha] \in \tilde{M}$, we have

$$D = \Phi_{[\alpha]} \circ \pi, \quad \text{where } \pi : \tilde{M} \to M.$$

since $\pi([\alpha]) = \alpha(1)$.

<u>Prop-n 3.10</u>: The developing map $D: \tilde{M} \to X$ satisfies:

1) For fixed basepoint x_0 and initial chart (U_0, φ_0), D is well-defined, independent of all other choices

2) D is a local diffeo-sm

3) Change of basepoint and initial chart gives a map equal to $g \circ D$ for some $g \in G$.

proof: (1) Exercise

(2) follows from (1) plus the fact that the $\gamma_{i-1,i}(x_i)$ and φ_{n-1} are local diffeos, and the topology on $\tilde{M}$.

(3) Exercise.

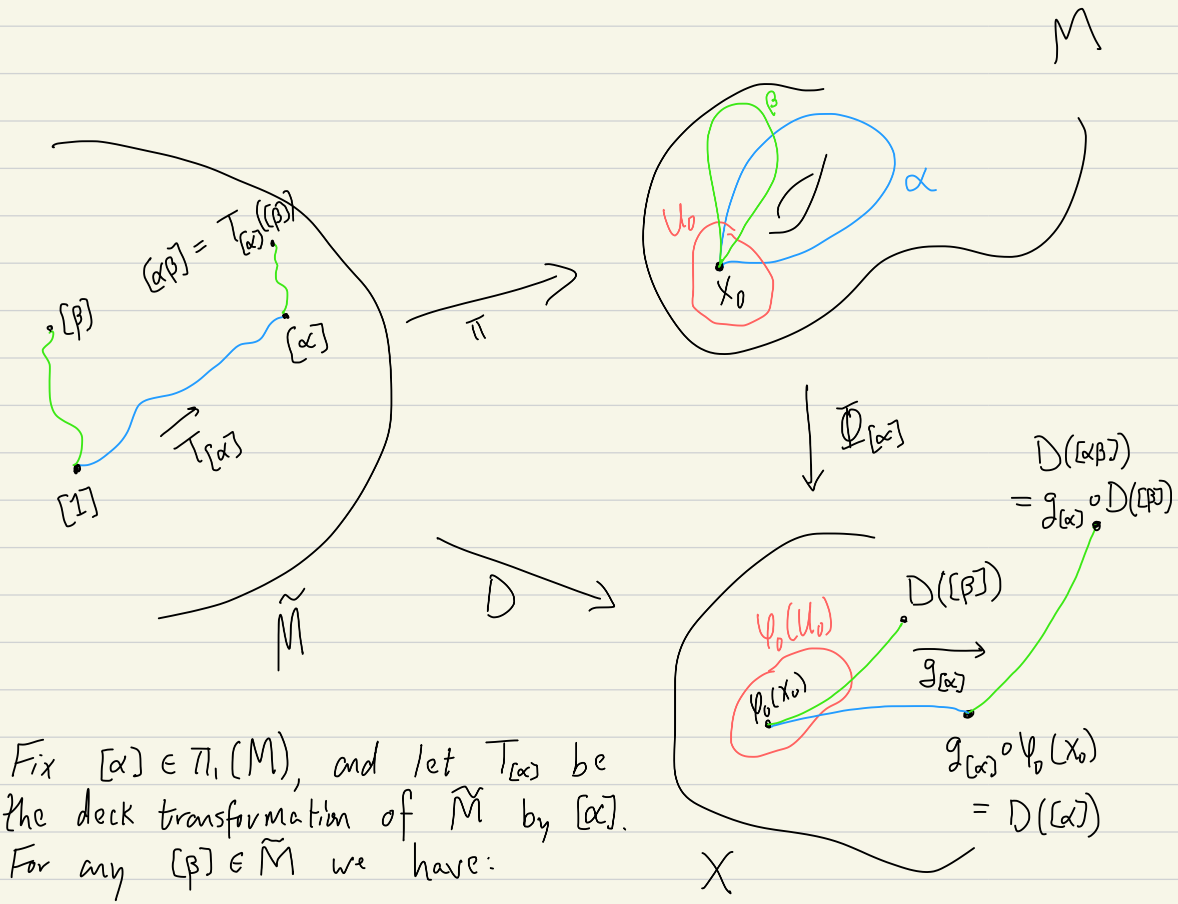
<u>Holonomy Map</u>:

Let $[\alpha] \in \tilde{M}$ be an element of $\pi_1(M)$ (i.e., its a closed loop in M).

Then $\Phi_{[\alpha]} : U \longrightarrow X$ where U is a nbhd. of the basepoint x_0.

Thus, φ_0 and $\Phi_{[\alpha]}$ are both charts in a nbhd. of x_0, so they differ in a nbhd. of x_0 by an element of G. Define $g_{[\alpha]} \in G$ by

$$\Phi_{[\alpha]} = g_{[\alpha]} \varphi_0$$

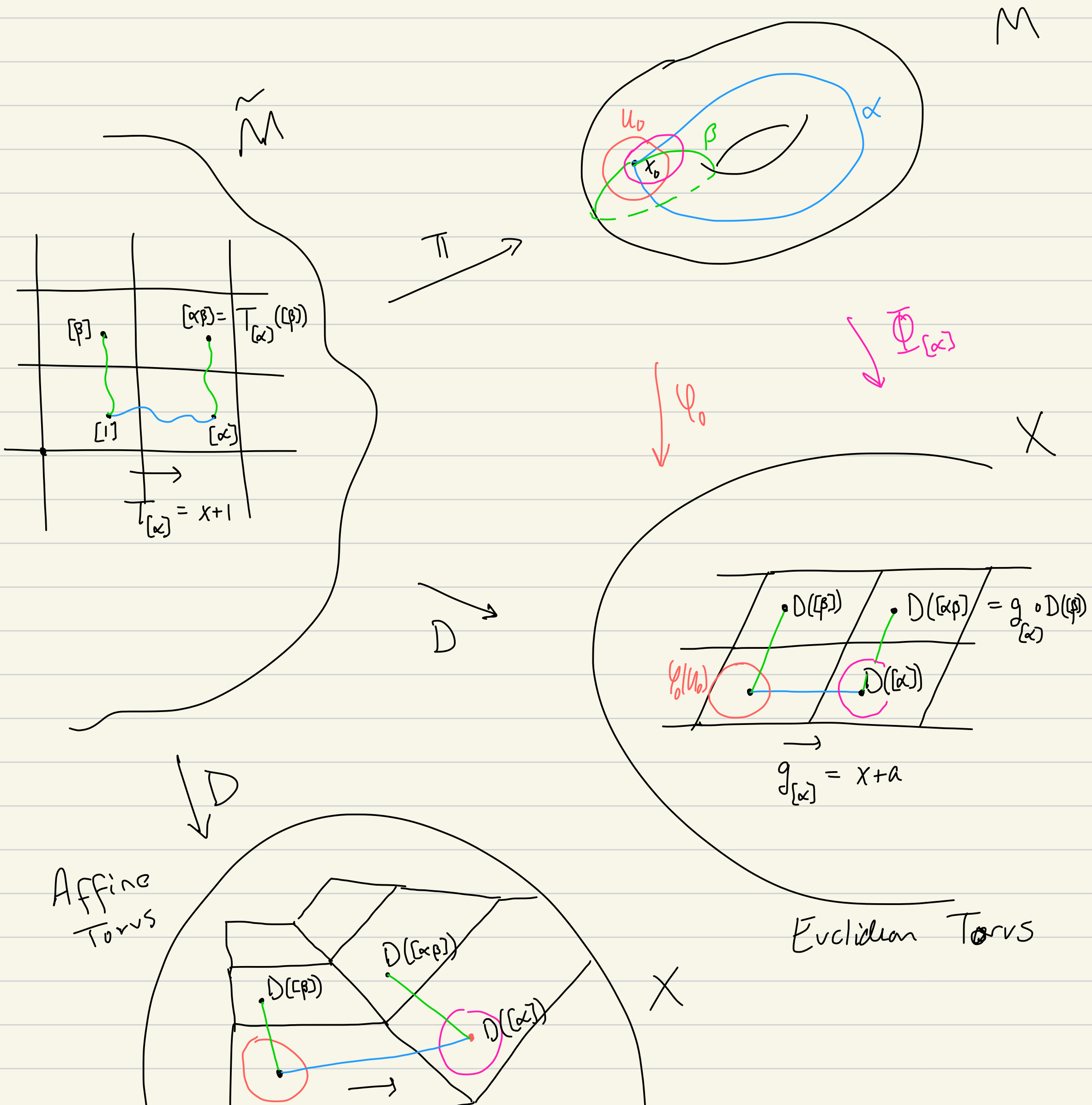


Fix $[\alpha] \in \pi_1(M)$, and let $T_{[\alpha]}$ be the deck transformation of $\tilde{M}$ by $[\alpha]$. For any $[\beta] \in \tilde{M}$ we have:

$$D \circ T_{[\alpha]}([\beta]) = D([\alpha\beta]) = g_{[\alpha]}(D([\beta])) = g_{[\alpha]} \circ D([\beta]), \text{ so}$$

$$D \circ T_{[\alpha]} = g_{[\alpha]} \circ D \quad \text{for all } [\alpha] \in \pi_1(M).$$

Ex: Euclidean Torus + Affine Torus



$g_{[\alpha]} = Ax + a$ where A rotates and scales.

For $[\alpha], [\beta] \in \pi_1(M)$, we have

$$g_{[\alpha\beta]} \circ D = D \circ T_{[\alpha\beta]} = D \circ T_{[\alpha]} \circ T_{[\beta]} = (g_{[\alpha]} \circ D) \circ T_{[\beta]}$$
$$= g_{[\alpha]} \circ (D \circ T_{[\beta]})$$
$$= g_{[\alpha]} \circ g_{[\beta]} \circ D$$

$$\therefore \quad g_{[\alpha\beta]} = g_{[\alpha]} \circ g_{[\beta]}$$

$\therefore$ The map $\rho : \pi_1(M) \longrightarrow G$ defined by

$$\rho([\alpha]) = g_{[\alpha]}$$

is a group homomorphism

Def-n: The elt $g_{[\alpha]}$ is the holonomy of $[\alpha]$. The group homomorphism ρ is called the holonomy of M Its image is the holonomy group of M.

Exercise: changing the basepoint x_0 and initial chart (U_0, φ_0) in the definition of G changes $\rho(\pi_1(M))$ by conjugation in G.

Def-n: M is a complete (G,X)-manifold if $D : \tilde{M} \longrightarrow X$ is a covering map.

- If M is complete and X is simply connected, then $D : \tilde{M} \longrightarrow X$ is a homeo-sm, so we can identify X with $\tilde{M}$.

Prop-n: If G is a group of real analytic diffeos. of a simply connected space X, and M is a complete (G,X)-mfld, then $M \cong X/\Gamma$ where Γ is the holonomy group of M.

proof: M complete and X simply connected
$\Rightarrow$ D a homeo-sm, so

$$g_{[\alpha]} = D \circ T_{[\alpha]} \circ D^{-1}$$

$\Rightarrow$ $\rho : [\alpha] \longmapsto g_{[\alpha]}$ is an iso-sm onto Γ

$$g_{[\alpha]} = g_{[\beta]} \Rightarrow D \circ T_{[\alpha]} \circ D^{-1} = D \circ T_{[\beta]} \circ D^{-1}$$

$$\Rightarrow T_{[\alpha]} = T_{[\beta]} \Rightarrow [\alpha] = [\beta]$$

$\therefore$ $\Gamma \cong \pi_1(M)$, so $M \cong \tilde{M}/\pi_1(M) \cong X/\Gamma$.

Prop-n: Let G be a Lie group acting analytically and transitively on a mfld X, s.t. the stabilizer G_x of x is compact for some (hence all) $x \in X$. Then X admits a G-invariant metric, and every closed (G,X)-mfld is complete.
proof: Thurston Prop 3.4.10 + Lemma 3.4.11

Thm 3.19: Let M be an n-mfld with a (G,X)-structure, where G acts transitively on X, and X admits a complete G-invariant metric. Then the following are equivalent:

(a) M is complete as a (G,X)-mfld.

(b) For some $\varepsilon > 0$, every closed ε-ball in M is compact.

(c) For every $a > 0$, every closed a-ball in M is compact

(d) There is some family of compact subsets S_t of M, for $t \in \mathbb{R}_+$, s.t.

$$\bigcup_{t \in \mathbb{R}_+} S_t = M \quad \text{and}$$

S_{t+a} contains a nbhd of radius a about S_t

(e) M is complete as a metric space.

$$\begin{tikzcd} \tilde{M} \arrow[r] \arrow[d] & M \arrow[dl] \\ X & \end{tikzcd}$$

proof: (a) $\Rightarrow$ (b) In general, for a cover $p: Y \to Z$ between Riemannian mflds s.t. p preserves the metric, we have

$$p(\bar{B}_\varepsilon(y)) = \bar{B}_\varepsilon(p(y))$$

since distances are defined in terms of paths, and path lift.

$\therefore$ ε-balls compact in Y $\iff$ ε-ball compact in Z

Choose a closed ε-ball in X that is compact; since G acts transitively, this ε works for all $x \in X$.

$\Rightarrow$ (b) holds for $\tilde{M}$ $\Rightarrow$ (b) holds for M.

(since $D: \tilde{M} \to X$ is a covering map) (since $\tilde{M} \to M$ is a covering map)

(b) $\Rightarrow$ (c) Induction: Suppose all closed a-balls are compact for some $a \geq \varepsilon$. Then $\bar{B}_a(x)$ can be covered w/ finitely many $\varepsilon/2$ balls, and so $\bar{B}_{a+\varepsilon/2}(x)$ can be covered w/ finitely many ε-balls, which are compact. $\therefore$ $\bar{B}_{a+\varepsilon/2}(x)$ is compact.

(c) $\Rightarrow$ (d) Let S_t be the ball of radius t about a fixed point.

(d) $\Rightarrow$ (e) Any Cauchy sequence is contained in some S_t, so it converges.

(e) $\Rightarrow$ (a) Suppose M is metrically complete.

- $\tilde{M}$ is metrically complete (w/ induced metric) since any Cauchy sequence in $\tilde{M}$ projects to one in M. Since M is complete the sequence has a limit x, and x has a compact nbhd. that lifts homeomorphically to $\tilde{M}$

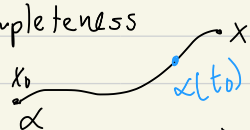
n.t.s.: $D : \tilde{M} \to X$ is a covering map.

For $x \in X$, consider $D^{-1}(x) \subseteq \tilde{M}$. If $D^{-1}(x) \neq \emptyset$ is discrete, we can find $\varepsilon > 0$ s.t. the open ε-balls centered at the elts. of $D^{-1}(x)$ are disjoint, and map homeomorphically to a ball about x.

If $D^{-1}(x) \neq \emptyset$ is not discrete, it contains a Cauchy sequence $\{y_n\}$. Since $\tilde{M}$ is complete $y_n \to y$ for some $y \in \tilde{M}$. since $D(y_n) = x \ \forall n$, $D(y) = x$. But the D cannot be a local homeo-sm at y, a contradiction. $\therefore$ D is a covering map onto its image. n.t.s. $D(\tilde{M}) = X$.
If $D^{-1}(x) = \emptyset$, the let $x_0 \in X$ be s.t. $D^{-1}(x_0) \neq \emptyset$. Let α be a path from x_0 to x. Let

$$t_0 = \sup\{t \in [0,1] \mid \alpha([0,t_0]) \text{ doesn't lift}\}$$

$\Rightarrow$ $\alpha([0,t_0))$ lifts to $\tilde{M}$. Completeness of $\tilde{M}$ $\Rightarrow$ $\alpha([0,t_0])$ lifts, a contradiction.



$\therefore$ $D : \tilde{M} \to X$ is onto. $\square$

Note: This proof avoids using that local homeo + path lifting prop $\Rightarrow$ covering map.

3.2.2 Completeness of hyperbolic polygonal gluings.

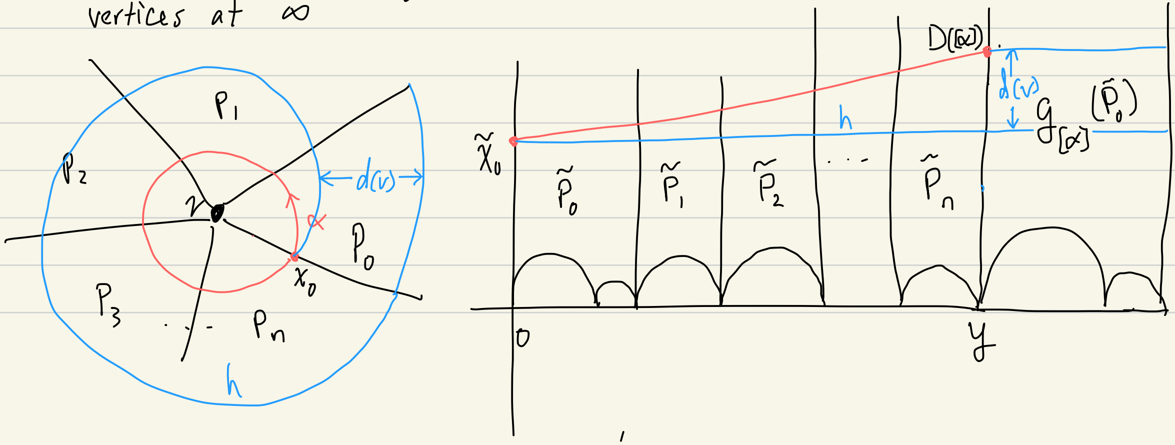
<u>Recall</u>: A gluing of hyp-c polygons has a hyp-c structure if the angle sum at each finite vertex is 2π.

- Completeness can only fail at <u>ideal vertices of M</u>

  ↳ ideal vertex: equivalence class of polygon vertices under gluing equivalence

- Let v be an ideal vertex of M. Let P_0 be a polygon with vertex v_0 in the class of v. Lift P_0 to a polygon $\tilde{P}_0$ in $\mathbb{H}^3$, with v lifting to ∞, another vertex at 0, and all other vertices on $\mathbb{R}_{>0}$.
- Let x_0 be a point on the left edge of P_0 (when facing v), and let α be a loop based at x_0 and going around v counterclockwise. Let $\tilde{x}_0$ be the lift of x_0 in $\tilde{P}_0$.

- As we develop along α, we add polygons $P_1, P_2, \ldots, P_n$ with vertices at ∞



- The developing image $D([\alpha])$ of $[\alpha] \in \tilde{M}$ is some point on the right edge of P_n, which has endpts. y and ∞

- Since the right edge of P_n glues to the left edge of P_0, $g_{[\alpha]}$ must be a hyperbolic isometry that takes 0 to y and $\tilde{x}_0$ to $D([\alpha])$.

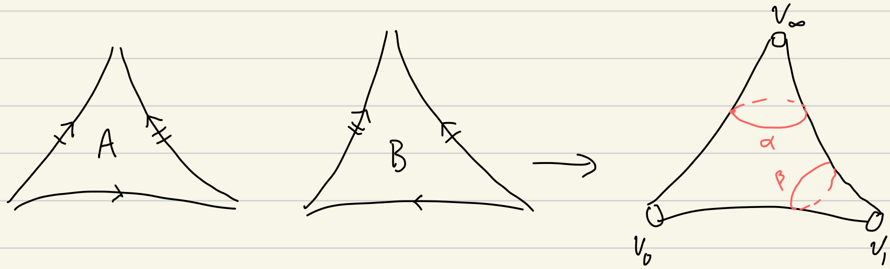
  Such $g_{[\alpha]}$ takes the horocycle through $\tilde{x}_0$ to the one through $D([\alpha])$.

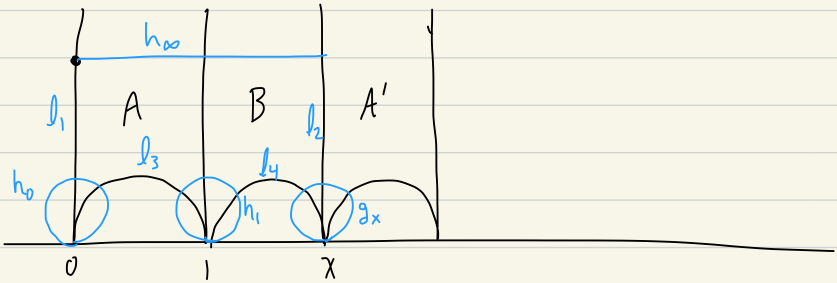
  Let $d(v)$ be the distance between these two horocycles.

<u>Prop-n 3.15</u>: Let M be a surface with hyp-c structure obtained by gluing hyperbolic polygons. Then the metric on M is complete $\iff$ $d(v) = 0$ for each ideal vertex v.

proof: stay tuned.

Example: Thrice-punctured sphere





- choose horocycles h_∞, h_0, h_1 about $0, 1, \infty$.

- $g_{[\alpha]}$ is the holonomy about the loop α around the vertex lifted to ∞, $g_{[\beta]}$ the holonomy about the vertex lifted to 1.
- complete structure $\Rightarrow$ $g_{[\alpha]}(h_\infty) = h_\infty$

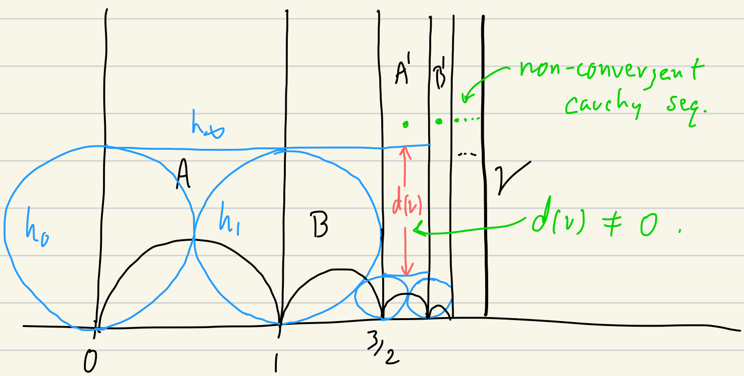
- g an isometry $\Rightarrow$ $l_1 = g(l_1) = l_2$, so h_0 and h_x have the same Euclidean diameter.

- similarly, the holonomy $g_{[\beta]}$ maps h_0 to h_x, so $l_3 = l_4$

$\therefore$ $x = 2$, and $g_{[\alpha]} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

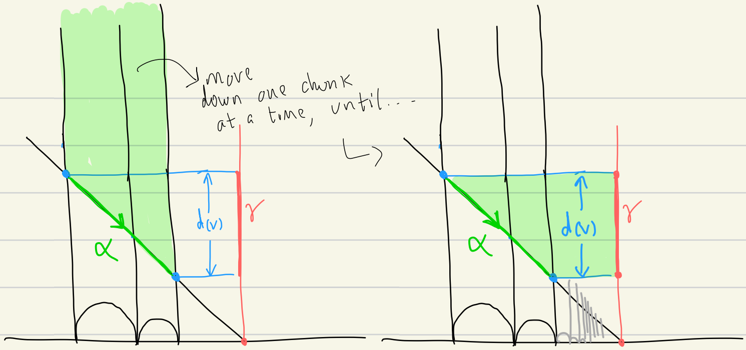
Prop-n: There is a unique complete hyperbolic structure on the 3-punctured sphere. A fundamental region for the structure is given by two ideal triangles with vertices $0, 1, \infty$ and $1, 2, \infty$, resp.

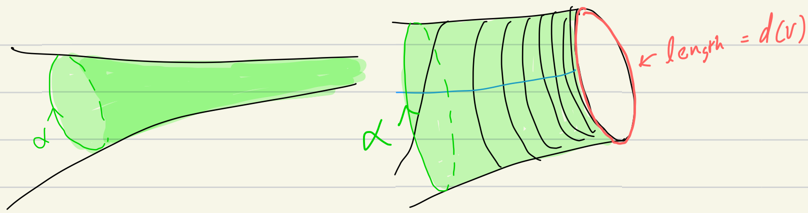
Incomplete Structure on  : For completeness we need $x=2$. What if we take $x = 3/2$?

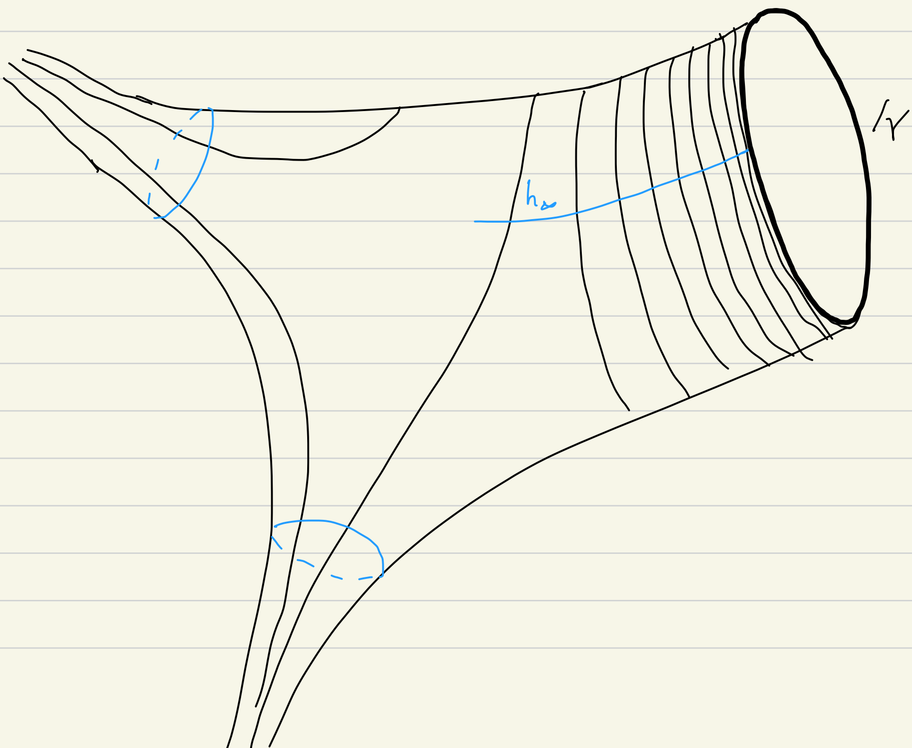


- choose h_∞ at height 1, and h_0, h_1 of (Euclidean) diameter 1.
- may assume horoball at $3/2$ has diameter <1 (otherwise consider α^{-1}). If the cusp at 1 is complete, the horoball at $3/2$ will be tangent to h_1. Either way, $g_{[\alpha]}(A \cup B)$ is narrower than $A \cup B$.
- Continue $\longrightarrow$ triangles limit to a vertical geodesic γ (why do they converge? geometric series).
- Can complete the structure by adding limit points. In $\mathbb{H}^2$ this corresponds to adding γ (and other translates) to the devel. image. In M, we add a geodesic loop (quotient of γ)

a better drawing:







Proof of Prop 3.15: Suppose $d(v) \neq 0$. Let h be a horocycle about v, and take a sequence of points along h, one for each edge crossed. This is a Cauchy sequence that does not converge.

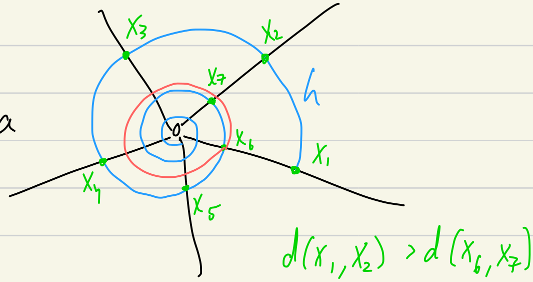
- Now suppose $d(v) = 0$ for all ideal vertices v. For each vertex v_i, let $h_i(t)$ be the open horoball about v_i of Euclidean radius t.

Let $S_t = M \setminus \left(\bigcup_i h_i(t) \right)$

Then $\bigcup S_t = M$, each S_t is compact, and

S_{t+a} contains a radius a nbhd. of S_t, so

$\Rightarrow$ Completeness



Another point of view:

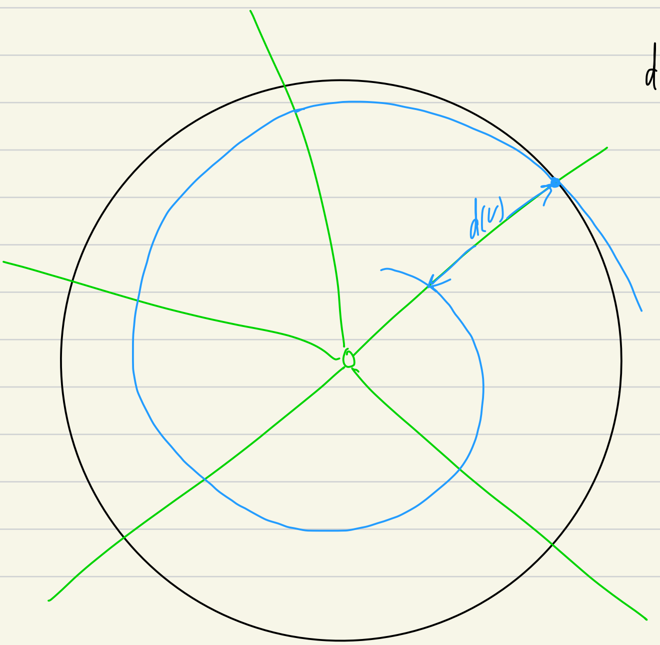
scaling, rotation, translation

$S^1 = lk(v)$ has a $\boxed{\text{similarity structure}}$.

- For each triangle T with ideal vertex at v, $lk(v) \cap T$ has a Euclidean structure coming from identification with the horosphere $\cap$-ing T.

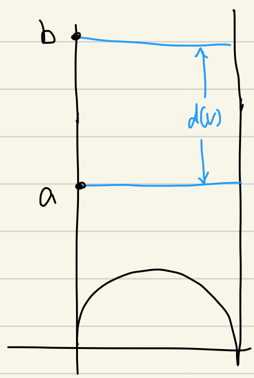
- If we go around v once, returning the starting triangle T, we'll be in a horosphere that differs from starting horosphere by $d(v)$.

- the map between these is a similarity: $\begin{pmatrix} e^{d(v)/2} & 0 \\ 0 & e^{-d(v)/2} \end{pmatrix}$



$d_{\mathbb{H}^2}(a,b) = \log(b-a) = d(v)$

$d_{\mathbb{E}^2}(a,b) = b-a = e^{d(v)}$



In n-dimensions, the picture is similar:

- in each polyhedron P with a vertex at v, $P \cap \mathrm{lk}(v)$ has a Euclidean structure from the horosphere cross section.
- Go around a loop in $\mathrm{lk}(v)$, come back to P, in a horosphere that differs from the starting one by a similarity.

<u>Theorem</u>: Let M be an n-mfld with hyperbolic structure obtained by gluing hyperbolic polyhedra. Then TFAE:

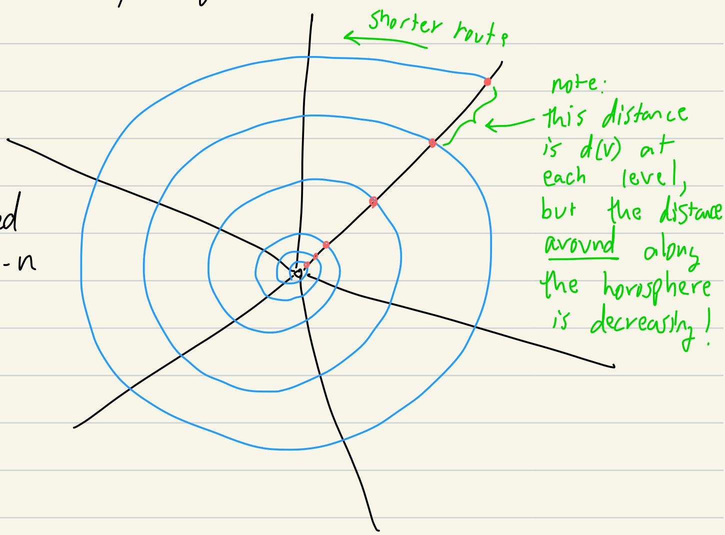
(a) M is complete

(b) For each ideal vertex v, the holonomy of $\mathrm{lk}(v)$ consists of Euclidean isometries

(c) For each ideal vertex v, $\mathrm{lk}(v)$ is complete as a similarity mfld.

<u>proof</u>: (b) $\Rightarrow$ (a) If holonomies are Euclidean isometries, then horosphere cross section match up if we go around a loop in $lk(v)$. So local cross sections glue up the give a global cross-section, a horoball nbhd. of v. $M \setminus \{\text{horoball nbhds.}\}$ is compact. Delete smaller & smaller horiball nbhds. to get an exhaustive family S_t of compact sets, apply 3.4.15 (d) $\Rightarrow$ (e)

(a) $\Rightarrow$ (b) If some holonomy around a loop α in $lk(v)$ is a contraction (if not an isometry must be a contraction in one direction or the other), then every time we go around v along a horosphere, the distance decreases (exponentially, in fact), so we get a Cauchy sequence that does not converge:



(b) $\Rightarrow$ (c)

$lk(v)$ is a closed mfld. Apply prop-n before Thm 3.19

$(c) \Rightarrow (b)$:

$lk(v)$ complete

$\Rightarrow$ can identify $\mathbb{E}^n$ with $\widetilde{lk(v)}$, and holonomy group of $lk(v)$ with deck transformations.

If some holonomy g is a contraction, then g has a fixed pt. x, and the orbit of g contains pts. limiting to x. Thus $p^{-1}(p(x))$ is not discrete, so $p : \mathbb{E}^n \to lk(v)$ cannot be a cover.

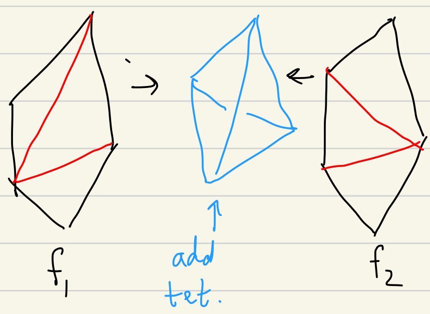
Chapter 4: Hyperbolic Structures + Triangulations

Defn: Let M be a 3-mfld.

- A (topological) ideal triangulation is a topological polyhedral decomposition such that all vertices are ideal and all polyhedra are tetrahedra.

- A geometric ideal triangulation is a topological ideal triangulation that is a geometric polyhedral decomposition.

– Given an ideal topological polyhedral decomposition, we can easily get an ideal triangulation.
  - cut each polyhedron into tetrahedra.
  - If cutting choices don't match between two glued polyhedral faces, add "flat" tetrahedra to interpolate:



The above method does not work for geometric ideal triangulations.

Open Q: Does every hyperbolic 3-mfld admit a geometric ideal triangulation

- See Purcell 4.1.1 for an extended example showing how to get a triangulation for a knot complement using the polyhedral decomposition.

4.2 Edge gluing equation

Let τ be a hyperbolic ideal tetrahedron

- Let e be an edge of τ.
- Can embed τ in $\mathbb{H}^3$ so that the endpts. of e are at 0 and ∞, and the other two vertices are at 1 and z, for some $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$.
- We'll say a tetrahedron as described above is in standard position w.r.t the edge e.

Def: The edge invariant $z(e)$ of e is the complex number z as described above.

<u>Rmk</u>: If $\mathrm{Re}(z(e)) = 0$ or if $z(e) = \infty$, then we say that T is:

- <u>degenerate</u> if $z \in \{0, 1, \infty\}$
- <u>flat</u> if $z \notin \{0, 1\}$

<u>Lemma</u>: Let T be a ideal tetrahedron with edge e_1, in standard position w.r.t. e_1. Then:

(1) The edge e_2 with vertices ∞ and 1 has edge invariant

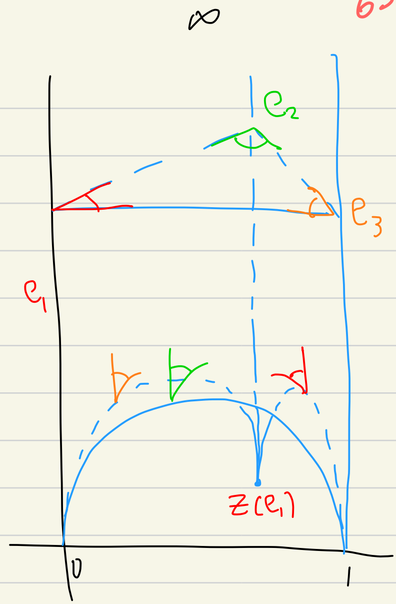
$$z(e_2) = \frac{1}{1 - z(e_1)}$$

(2) The edge e_3 with vertices ∞ and $z(e_1)$ has edge invariant

$$z(e_3) = \frac{z(e_1) - 1}{z(e_1)}$$

(3) The edge e_i' opposite e_i has edge invariant $z(e_i)$.

- Thus, $z(e_1) \cdot z(e_2) \cdot z(e_3) = -1$, $1 - z(e_1) + z(e_1) z(e_3) = 0$



Proof: Set $z = z(e_1)$.

(1) Let $T : w \longmapsto \dfrac{w-1}{z-1}$

then T maps $\infty \mapsto \infty$, $1 \mapsto 0$, $z \mapsto 1$, and $0 \mapsto \dfrac{-1}{z-1}$.

$\therefore\ z(e_2) = T(0) = \dfrac{1}{1-z}$

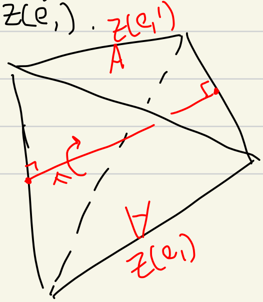
(2) Let $T : w \longmapsto \dfrac{w-z}{-z}$

the T fixes ∞, and maps $z \mapsto 0$, $0 \mapsto 1$, and $1 \longmapsto \dfrac{(1-z)}{-z}$

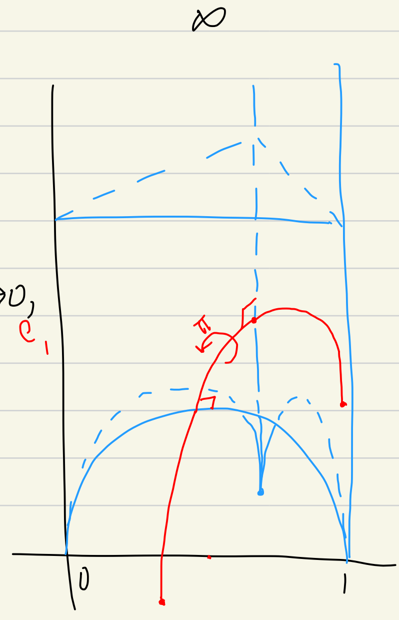
$\therefore\ z(e_3) = T(1) = \dfrac{z-1}{z}$.

3) For e_1 (others similar).

Let γ be the unique geodesic meeting e_2 and e_2'. Rotation by π about γ is an isometry taking $0 \mapsto 1$, $1 \mapsto 0$, $z(e_1) \mapsto \infty$, $\infty \mapsto z(e_1)$.



$\therefore\ z(e_1') = z(e_1)$



Now consider a gluing of ideal tetrahedra $T_1, \ldots, T_k$ around an edge e.

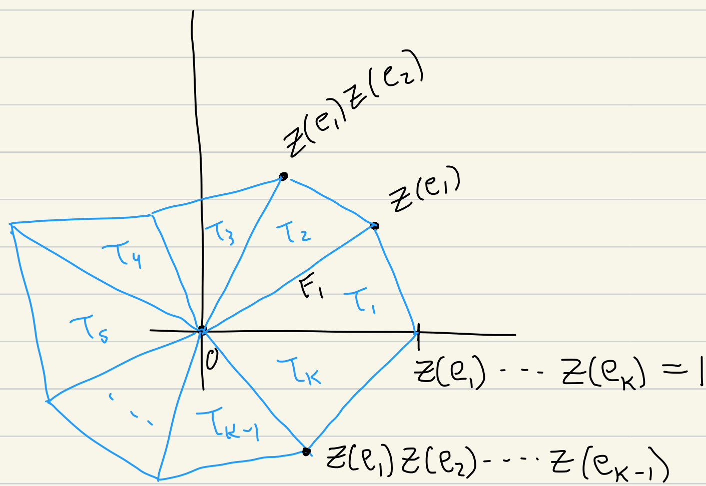
- Let e_i be the edge of T_i glued to e.

- Put T_1 in standard position w.r.t. e_1, and let F_1 be the face of T_1 with vertices $0, \infty, z(e_1)$.

- F_1 glues to some face F_1' in a tetrahedron T_2 with edge e_2 glued to e. First put T_2 in standard position w.r.t. e_2, then apply an isometry of $\mathbb{H}^3$ fixing 0 and ∞ and taking $1 \mapsto z(e_1)$. This takes the fourth vertex of T_2 to $z(e_1)z(e_2)$

orientation preserving !

- Continue in this way, attaching $T_3, \ldots T_k$.



<u>Theorem 4.7</u> (Edge Gluing Equations): Let M^3 admit a topological ideal triangulation s.t each tetrahedron has a hyperbolic structure and gluing maps are isometries. The hyp-c structures on the tetrahedra induce a hyperbolic structure on M if and only if for each edge e of M

"edge gluing equations" $\rightarrow$ $$\left[\prod z(e_i) = 1 \qquad \sum \arg\big(z(e_i)\big) = 2\pi\right]$$ (# unknowns = # tets.)

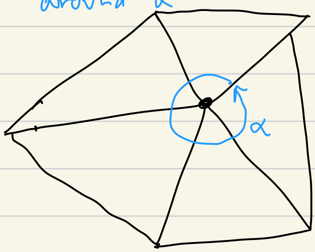
with product/sum over all edges gluing to e_i.

<u>Proof</u>: hyp-c structure on tetrahedra induces hyp-c structure on M $\Leftrightarrow$ every point in M has a nbhd isometric to a ball in $\mathbb{H}^3$. If a point on a edge has a ball nbhd., then the angle around the edge must be 2π, so $\sum \arg(z(e_i)) = 2\pi$. Also, there cannot be a non-trivial translation as we move around the edge, i.e., the holonomy should be trivial, i.e., faces should match up.

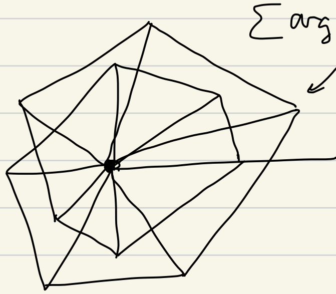
$\Rightarrow \prod z(e_i) = 1$

Converse is clear.

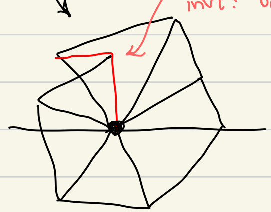
non-trivial holonomy around α



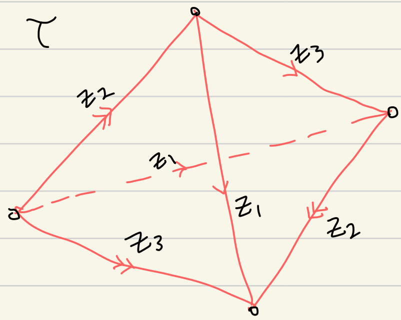
$\sum \arg(z_i) \neq 2\pi$



negative edge invt! bad!

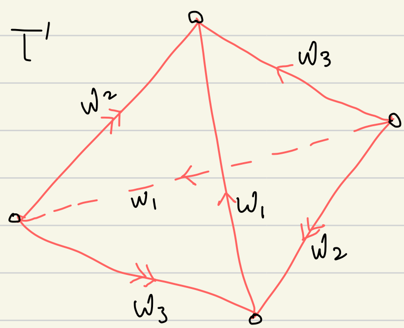


Example: fig-8 knot.



- Recall: we have a decomp. of the fig-8 knot complement $S^3 \setminus K$ into a pair of ideal tetrahedra.

Goal: Find edge invts. so that the gluing of tetrahedra is geometric — i.e., so that the gluing induces a complete hyp-c metric on $S^3 \setminus K$.



- Let z_i and w_i be edge invts. for the tetrahedra, $i=1,2,3$.

- Edge gluing equations:

$$\circ\!\!\rightarrow\!\!\circ : \quad z_1^2 z_3 w_1^2 w_3 = 1 ; \qquad \circ\!\!\twoheadrightarrow\!\!\circ : \quad z_2^2 z_3 w_2^2 w_3 = 1$$

- set $z = z_1$, $w = w_1$, and recall that

$$z_3 = \frac{z-1}{z} \; ; \quad w_3 = \frac{w-1}{w}$$

so for $\circ\!\!\rightarrow\!\!\circ$ we get $z^2\left(\frac{z-1}{z}\right) w^2 \left(\frac{w-1}{w}\right) = 1$

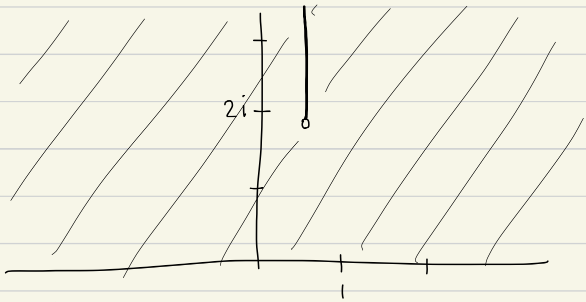
$$\Rightarrow \; z(z-1)w(w-1) = 1 \qquad (*)$$

$$\Rightarrow \; z = \frac{1 \pm \sqrt{1 + 4/(w(w-1))}}{2}$$

Rmk: easy the check that the equation for 
gives the same result.

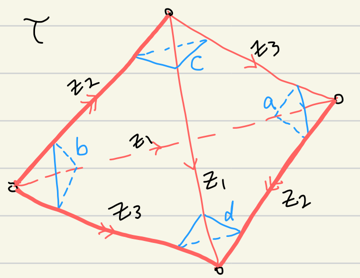
need: $\operatorname{Im}(z) > 0$, so need $1 + \frac{4}{w(w-1)} < 0$. This holds for

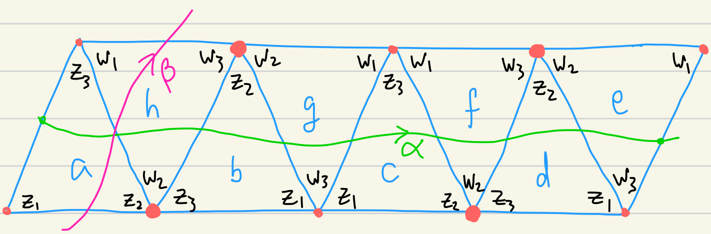
$$w \in \mathbb{C} \setminus \left\{ x+iy \;\middle|\; x = \tfrac{1}{2},\ y > \tfrac{\sqrt{15}}{2} \right\}$$



- Let T be the cusp torus of $S^3 \setminus K$. T is made up of triangle cross-sections of corners of tetrahedra, which fit together as follows:

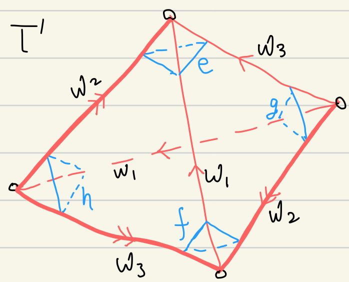
τ





looking from inside M..

T'



- holonomies of α, β are translations

$\Longleftrightarrow$

$$z_3 w_2^{-1} z_2 w_3^{-1} z_3 w_2^{-1} z_2 w_3^{-1} = 1 \qquad \text{and} \qquad w_1 z_2^{-1} = 1$$

$$\Longleftrightarrow \left(\frac{z_2 z_3}{w_2 w_3}\right)^2 = 1 \quad \text{and} \quad \frac{w_1}{z_2} = 1$$

$$\Longleftrightarrow \left(\frac{1}{1-z}\,\frac{z-1}{z}\,\frac{1-w}{1}\,\frac{w}{1-w}\right)^2 = \left(\frac{w}{z}\right)^2 = 1 \quad \text{and} \quad w(1-z)=1$$

$$\Longleftrightarrow w = \pm z \quad \text{and} \quad \boxed{z(1-z) = 1}$$

not possible since need $\operatorname{Im}(w) > 0$, $\operatorname{Im}(z) > 0$.

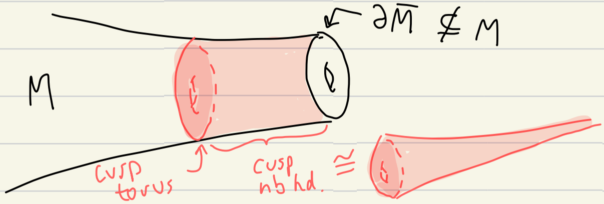
by $(*)$,
$$z = \frac{1 + \sqrt{1 + 4/(w(w-1))}}{2}$$
$$= \frac{1 + \sqrt{1 + 4/z(z-1)}}{2}$$
$$= \frac{1 + \sqrt{1 + 4/(-1)}}{2} = \frac{1 + \sqrt{-3}}{2}$$

$$\therefore z = w = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$\therefore$ The fig-8 knot complement has a unique complete structure.

4.3: Completeness Equations:

Def-n: Let $\bar{M}$ be a 3-mfld with torus boundary components. Let $M = \text{int}(\bar{M})$ be the interior. a cusp (or cusp neighborhood) of M is a closed nbhd. of a component of $\partial\bar{M}$ in M. A cusp cross-section (or cusp torus) is a bdy component of $M \setminus \{\text{cusps}\}$. We say M is a mfld. with torus cusps.



Rmk: If M^3 is a $(\text{Isom}^+(\mathbb{H}^3), \mathbb{H}^3)$-mfld with torus cusps, then the metric on M is complete $\iff$ the hyp-c structure on M induces a Euclidean structure on the cusp tori. (Our proof was in the context of polyhedral gluings, but can be adapted).

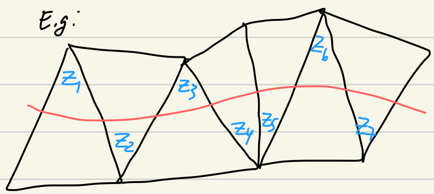
If M^3 is a mfld with torus cusps, then a (topological) ideal triangulation of M induces an cusp triangulation of each cusp torus.

Definition: Let M be a 3-mfld obtained as a gluing of hyp-c tetrahedra, and let T be a cusp torus of M. Let $[\alpha] \in \pi_1(T)$ so that α is a loop on T in the class of $[\alpha]$. Orient α on T and homotope so that α is a normal curve with respect to the cusp triangulation (i.e., Δ). Let $z_1, \dots, z_K$ be the edge invt. of the corner of triangles cut of by α, and let $\varepsilon_i = 1$ (resp. $\varepsilon_i = -1$) if z_i is to the left (resp. right) of α. Define

$$H([\alpha]) = \prod_{i=1}^{n} z_i^{\varepsilon_i}$$

Note: If $g_{[\alpha]}$ is the holonomy of α w.r.t. $D: \tilde{M} \to \mathbb{H}^3$, then $H([\alpha]) = g'_{[\alpha]}$

E.g.:



$H([\alpha]) = z_1 z_2^{-1} z_3 z_4^{-1} z_5^{-1} z_6 z_7^{-1}$

Prop-n 4.15 (completeness equations): Let M be a 3-mfld obtained by gluing ideal hyp-c tetrahedra, s.t. the edge gluing equations are satisfied. Let $T_1, \dots, T_K$ be cusp tori of M, and let α_i, β_i generate $\pi_1(T_i)$.

If $H([\alpha_i]) = H([\beta_i]) = 1$ for $i = 1, \dots, K$, then the triangulation is geometric and induces a complete structure on M.

proof: Exercise (recall fig-8 example).

6.2: Completion of Incomplete Structures. (we'll come back to ch. 5)

- For the fig-8, 1-$\mathbb{C}$ parameter family of incomplete structures — how do we make sense of completion of such structures?

- In 2-dim case, adjoin a loop consisting of limit pts

Let M^3 be a mfld with torus cusps, and incomplete hyperbolic structure. Let C be a cusp, and let T be a cusp torus for C (i.e, a cross-section).
M not complete $\Rightarrow$ similarity structure on T <u>not</u> Euclidean (for some cusp, let's assume it's T).

Note: from now on we'll drop the brackets on $[\alpha]$

Let α, β generate $\pi_1(T) \cong \mathbb{Z}\times\mathbb{Z}$, and let g_α and g_β be the holonomies of α and β w.r.t. $D: \tilde{M} \to \mathbb{H}^3$. Thus g_α and g_β commute.

- <u>Exercise:</u> If $g, h \in \mathrm{Isom}^+(\mathbb{H}^3)$ commute and are not both order-2, then either
  (1) g and h are parabolic with a common fixed point, or
  (2) g and h are loxodromic/elliptic + have the same axis.

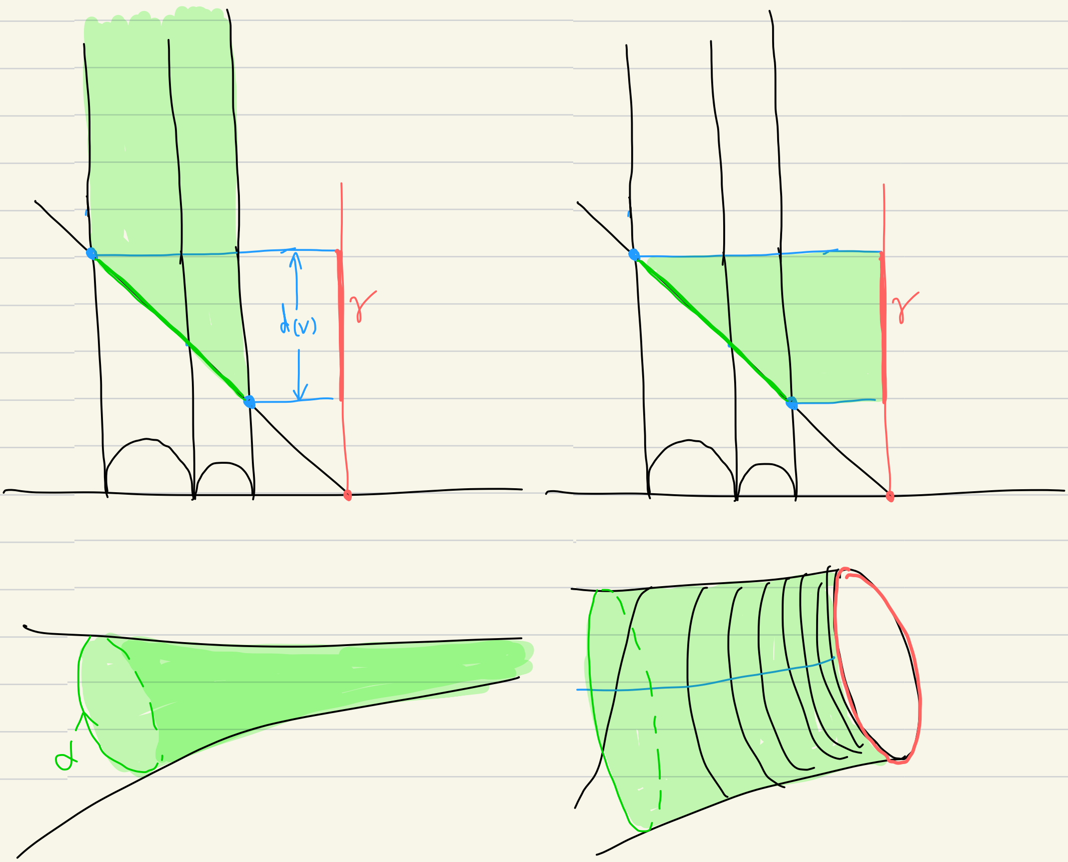
In case (1), g_α and g_β are translations, so the cusp is complete. So we must have (2).

We may assume the axis of g_α and g_β is the geodesic from 0 to ∞.

Let $N(C)$ be the cusp neighborhood for C so that T is the torus boundary of $N(C)$.

- Since $M \setminus \bigcup_{C_i \text{ a cusp}} N(C_i)$ is compact, we only need to complete cusp neighborhoods to complete M.
- Need to understand the developing image of $N(C)$.

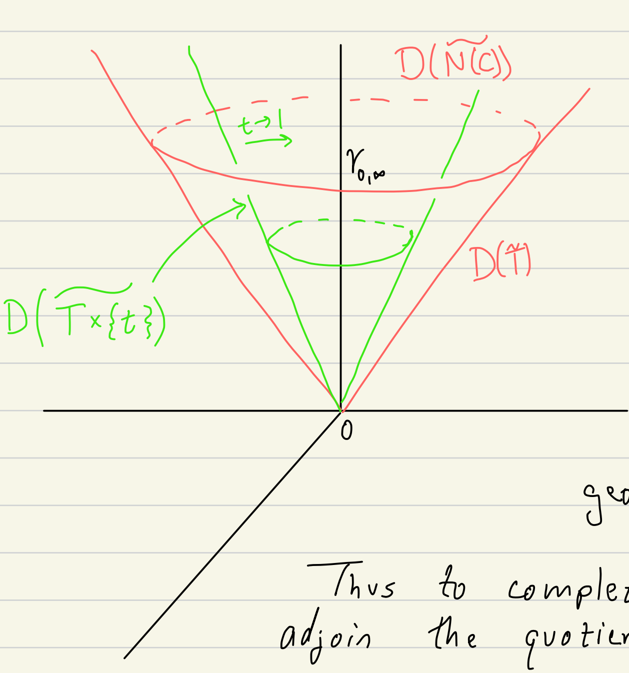
Recall, in 2-dims:

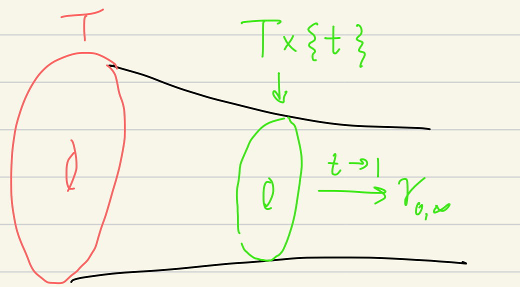


We have: $N(C) \cong T \times [0,1)$.

Since $\langle g_\alpha, g_\beta \rangle = \pi_1(T)$, they must fix the developing image of T in $\mathbb{H}^3$. It is easy to see that g_α, g_β cannot both be elliptic, so the only 2d subspaces fixed by $\langle g_\alpha, g_\beta \rangle$ are circular cones based at O.
invt. bananas

It follows that the developing image of $N(C)$ is a solid cone based at O, minus the geodesic $\gamma_{0,\infty}$. The boundary of this cone is the developing image of T.





To complete the developing image of $N(C)$, we need to adjoin the geodesic $\gamma_{0\infty}$ to $D(\widetilde{N(C)})$.

Thus to complete $N(C)$, we'll need to adjoin the quotient of $\gamma_{0,\infty}$ by $\langle g_\alpha, g_\beta \rangle$ to $N(C)$.

Start on board projector screen blocks

<u>Prop-n 6.4</u>: The completion of $N(C)$ is either homeomorphic to the 1-point compactification of $N(C)$ obtained by crushing $T\times\{1\}$ to a point, or it is homeo-c to a solid torus obtained by attaching a solid torus to $N(C)$ along $T\times\{1\}$.

<u>proof</u>: Let z be a point on $\gamma_{0,\infty}$. Then the orbit of z under $\langle g_\alpha, g_\beta\rangle$ is either a discrete subset of $\gamma_{0,\infty}$, or a dense subset.

If the orbit of z is dense, then the completion is the 1-point compactification, which is not a mfld. (exercise).

If the orbit of z is discrete, then the quotient of $\gamma_{0,\infty}$ is a closed curve whose length is the distance between orbit pts.

Let $\overline{N(C)}$ be the completion. A closed nbhd. of the attached geodesic is a solid torus, and removing this gives a mfld homeo-c to $N(C)$. Thus $\overline{N(C)}$ is obtained by attaching a solid torus along $T\times\{1\}$. $\square$

Rmk: The holonomies g_α and g_β are holonomies of α and β w.r.t. $D: \tilde{M} \to \mathbb{H}^3$. If we consider α and β w.r.t. $D: \tilde{T} \to \mathbb{R}^2$, then their holonomies h_α and h_β will be similarities, and tranlate non-trivially. More precisely, if h_α is the map $z \mapsto re^{i\theta}z + b$, then g_α is $z \mapsto re^{i\theta}z$.

Def-n: Let $\mathcal{C}$ be the solid cone about the geodesic $\gamma_{0,\infty}$ from 0 to ∞ with vertex at 0.

Let $\tilde{\mathcal{C}}$ be the infinite cyclic branched cover of $\mathcal{C}$ branched along $\gamma_{0,\infty}$, and let $\mathcal{C}_\alpha$ be the quotient of $\tilde{\mathcal{C}}$ by a rotation by an angle α.

A nbhd. of a point on $\gamma_{0,\infty}$ in $\mathcal{C}_\alpha$ is called a hyperbolic cone, with cone angle α.

A 3-dim. hyp-c cone mfld is a mfld M s.t. each point has a nbhd. isometric to either a ball in $\mathbb{H}^3$ or a hyp-c cone.

The points with cone nbhds. form a geodesic link in M, and are called the singular locus.

Let $I = \mathbb{C} \setminus \gamma_{0,\infty}$, and let $\tilde{I} = \tilde{\mathbb{C}} \setminus \gamma_{0,\infty}$ be the universal cover of I.

Thus $I = D(\widetilde{N(C)})$ and $\partial I = D(\tilde{T})$, and $\tilde{I} = \widetilde{N(C)}$.

The developing map $D: \widetilde{N(C)} \longrightarrow I$ lifts to $\tilde{I}$, and for $\alpha \in \pi_1(T)$ there is an associated holonomy $\tilde{g}_\alpha$ w.r.t. $\tilde{D}: \widetilde{N(C)} \longrightarrow \tilde{I}$.

$$\begin{tikzcd} & \tilde{I} \arrow[d] \\ \widetilde{N(C)} \arrow[ur, dashed, "\tilde{D}", "?"' near end] \arrow[r, "D"'] & I \end{tikzcd}$$

Show similarity structure visualization to demo.

What $\theta > 2\pi$ looks like

Why lift to $\tilde{I}$? We want to be able to consider elements of $\pi_1(T)$ that rotate about $\gamma_{0,\infty}$ by more than 2π. g_α only sees rotation mod 2π.

Isometries of $\tilde{I}$ are parametrized by $d + i\theta \in \mathbb{C}$, where d is the translation distance along $\gamma_{0,\infty}$, and θ is the angle of rotation.

Note: If $\theta \leq 2\pi$ and $g_\alpha : z \mapsto re^{it}z$, then $\tilde{g}_\alpha = \log(g_\alpha) = r + it$.

- In purcell, $\tilde{g}_\alpha$ is denoted $\mathcal{L}(\alpha)$ (see p.90), and is called the complex length of α.

In Thurston $\tilde{g}_\alpha$ is denoted $\tilde{H}(\alpha)$. I'm sorry.

Prop-n 6.8: When the completion $\overline{M}$ of M is a mfld., it is a hyp-c cone mfld, and the singular locus consists of geodesics attached in the completion.

proof: Let T, $N(C)$, $\gamma_{0,\infty}$, be as before.

define a homomorphism $\varphi: \pi_1(T) \longrightarrow \mathbb{R}_{>0}$

by $\gamma \longmapsto Re(\tilde{g}_\alpha)$, where $\tilde{g}_\gamma$ is the holonomy of γ w.r.t. $\tilde{D}: \widetilde{N(C)} \longrightarrow \tilde{I}$. Note that $Re(\tilde{g}_\alpha)$ is the translation distance of $\tilde{g}_\alpha$ along $\gamma_{0,\infty}$.

Show various similarity structure examples. what is the kernel? image?

Let α_1 generate $Ker\varphi$, and let α_2 be s.t. $\overline{\alpha}_2$ generates $\pi_1(T)/Ker\varphi$.

Let $d_2 + i\theta_2 = \tilde{g}_{\alpha_2}$. Then $s: d \mapsto \frac{d}{d_2} \cdot \alpha_2$ is a homo-sm satisfying $\varphi \circ s = 1$, so the sequence $1 \to Ker\varphi \longrightarrow \pi_1(T) \to Im\varphi \longrightarrow 1$ splits

$$\therefore \quad \pi_1(T) \cong Ker\varphi \times Im\,\varphi \quad \cong Ker\varphi \times \pi_1(T)/Ker\varphi$$

Upshot: with this new basis, it is easy to see what $\overline{N(C)}/\rho(\pi_1(T))$ is, since $\tilde{g}\alpha_1$ just

holonomy rep.

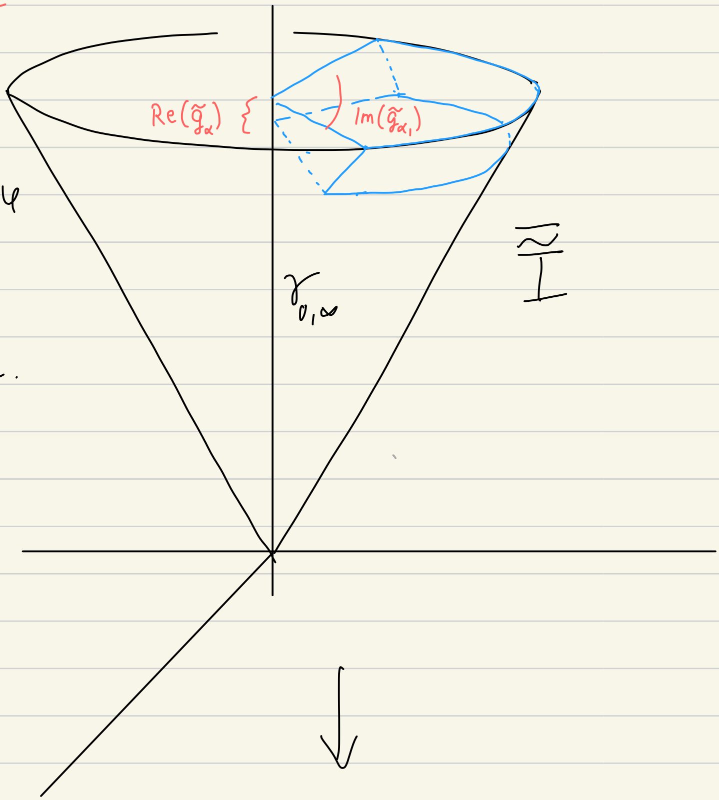
acts by rotation, and $\tilde{g}\alpha_2$ acts by translation and rotation along $\gamma_{0,\infty}$.

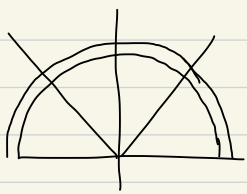
$$\varphi : \gamma \longmapsto \underbrace{Re(\tilde{g}_\gamma)}_{\text{translation distance of } \gamma}$$

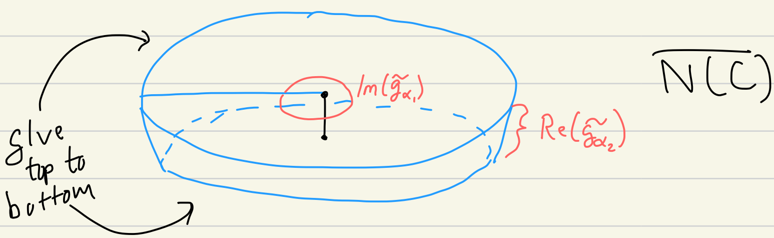
$$\pi_1(T) \cong \operatorname{Ker}\varphi \times \pi_1(T)/\operatorname{Ker}\varphi$$

$$\cong \langle \alpha_1 \rangle \times \langle \tilde{\alpha}_2 \rangle$$

$\uparrow$ elliptic $\qquad$ $\uparrow$ loxodromic.







$\therefore$ $\bar{M}$ is a cone mfld., and cone angle at completion points is $Im(\tilde{g}_{\alpha_1})$, core geodesic length is $Re(\tilde{g}_{\alpha_2})$. $\square$

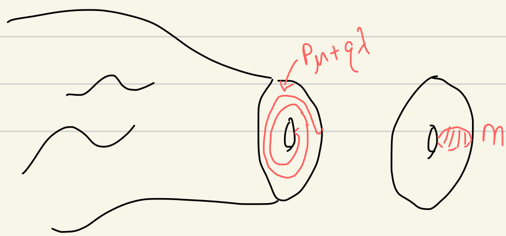
6.3 Hyperbolic Dehn filling space

Def-n: Let M be a mfld with torus bdy component T. Let μ, λ be a basis for $H_1(T;\mathbb{Z})$. Let $(0,0) \neq \underbrace{(p,q)}_{\text{this is a "slope" on } T} \in \mathbb{Z}\times\mathbb{Z}$.

- If $\gcd(p,q)=1$:
  We obtain the (p,q) Dehn filling of M along T by gluing a solid torus $\mathbb{T}$ to T so that the meridian of $\mathbb{T}$ glues to the curve $p\mu + q\lambda$ on T.
  "Dehn filling coefficients"

- If $\gcd(p,q)=d\neq 1$:
  Let $\mathbb{T}_d$ be the d-fold cyclic quotient of $\mathbb{T}$ by the symmetry that fixes the core curve of $\mathbb{T}$, and let m_d be the image of the meridian of $\mathbb{T}$ under this quotient.
  We obtain the (p,q) (orbifold) Dehn filling of M along T be gluing $\mathbb{T}_d$ to M along T so that m_d glues to $\frac{p}{d}\mu + \frac{q}{d}\lambda$.

- Note that when $\gcd(p,q)=d$, the (p,q) (orbifold) Dehn filling is homeomorphic to the $(\frac{p}{d}, \frac{q}{d})$ Dehn filling (but <u>not</u> isometric).



<u>Rmk</u>: If $M = S^3 \setminus L$ is a link complement, then there is a canonical choice of basis on each torus cusp T: choose μ to be a meridian of the link component L_T corresponding to T, and choose λ to be the homological longitude of T (this is the bdy of the homologically non-trivial surface in $S^3 \setminus L_T$).

For M a mfld with cusps $C_1, \dots, C_k$, denote by $M_{(p_1,q_1),\dots,(p_k,q_k)}$ the mfld obtained by (p_i,q_i) Dehn filling of the cusp C_i, $i=1,\dots,k$.

If (p_i,q_i) is replaced by ∞, then the cusp is unfilled. So $M = M_{\infty,\dots,\infty}$.

· while we're here...:

<u>Thm</u> (Wallace '60, Lickorish '62): Let M be a closed, orientable 3-mfld. Then M is obtained by Dehn filling the complement of a link in S^3.

· As defined, Dehn filling is a topological operation. By considering completion of hyp-c structures, we can understand Dehn filling geometrically.

(Assume that the completion is not 1-pt. compactification).

Definition: Given a basis $\mu, \lambda \in \pi_1(T)$ for a cusp torus T, the generalized Dehn filling coefficients (a, b) for $\overline{M}$ are solutions to the equation

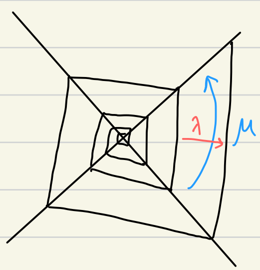
$$a\tilde{g}_\mu + b\tilde{g}_\lambda = 2\pi i$$

or $(a,b) = \infty$ if T is complete.

- In general, $a, b \in \mathbb{R}$, and $a\mu + b\lambda \in H_1(T, \mathbb{R})$
- If $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ are primitive $(\gcd(a,b) = 1)$, then

$$a\mu + b\lambda \text{ generates } \operatorname{Ker}(\alpha \mapsto \operatorname{Re}(\tilde{g}_\alpha)) = \operatorname{Ker}\varphi ,$$

so this filling corresponds to a completion that gives a mfld, i.e., the cone angle at the completion points is 2π.



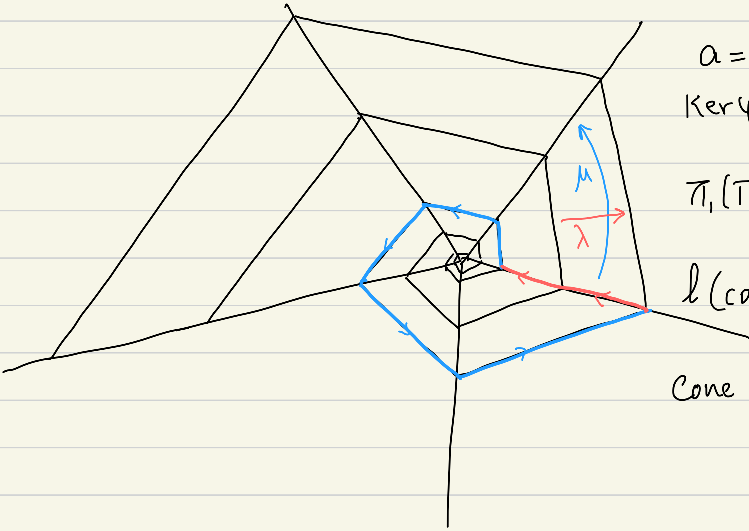
$a = 4, \; b = -1$

$\operatorname{Ker}\varphi = \langle 4\mu - \lambda \rangle$

$\pi_1(T)/_{\operatorname{Ker}\varphi} = \langle \overline{\mu} \rangle$

Core curve has length $\operatorname{Re}(\tilde{g}_\mu)$

Cone angle $= \operatorname{Im}(\tilde{g}_{4\mu - \lambda}) = 2\pi$



$a = 5, \quad b = -2$

$\operatorname{Ker}\varphi = \langle 5\mu - 2\lambda \rangle$

$\pi_1(T)/\operatorname{Ker}\varphi = \langle \overline{\lambda - 2\mu} \rangle$

$\ell(\text{core}) = \operatorname{Re}(\tilde{g}_{\lambda - 2\mu})$

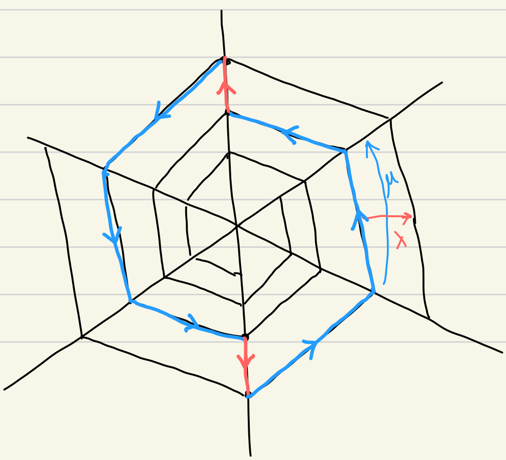
$\text{Cone angle} = \operatorname{Im}(\tilde{g}_{5\mu - 2\lambda})$

- If $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ and $\gcd(a,b) = d$, then

$$\frac{a}{d}\mu + \frac{b}{d}\lambda \quad \text{generates } \operatorname{Ker}\varphi$$

and the filling corresponds to a completion that gives an orbifold with cone angle $\frac{2\pi}{d}$ at completion points.

cone mfld. w/ cone angle $2\pi/K$, $K \in \mathbb{Z}$.



$a = 6, \quad b = 2 \qquad (d = 2)$

$\operatorname{Ker}\varphi = \langle 3\mu + \lambda \rangle$

$\pi_1(T)/\operatorname{Ker}\varphi = \langle \overline{\mu} \rangle$

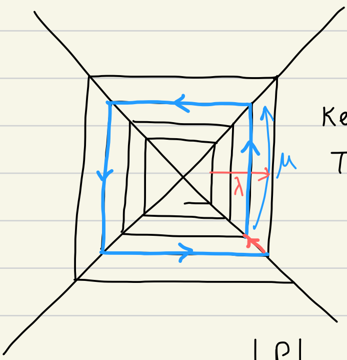
$\ell(\text{core}) = \operatorname{Re}(\tilde{g}_{\mu})$

$\text{cone angle} = \operatorname{Im}(\tilde{g}_{3\mu + \lambda})$

More generally:

- If $(a,b) \in \mathbb{Q} \times \mathbb{Q}$, then if $\frac{a}{b} = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$, then if $d = \left|\frac{p}{a}\right|$, the cone angle is $2\pi d$.

When $d > 1$, this does not correspond to a Dehn filling (with our def-n). (though one could allow gluing in cyclic branched cover of solid tori...)



$a = 4 \quad b = -\frac{1}{3}$

$\ker\varphi = \langle 12\mu - \lambda \rangle$

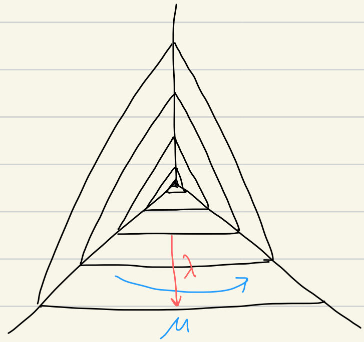
$\pi_1(T)/\ker\varphi = \langle \bar{\mu} \rangle$

$\frac{a}{b} = \frac{4}{-1/3} = \frac{-12}{1} = \frac{p}{q}$

$\left|\frac{p}{a}\right| = \frac{12}{4} = 3$

$\Rightarrow$ cone angle $= 6\pi$

$\ell(\text{core}) = \operatorname{Re}(\tilde{g}_\mu)$.



$a = 3 \quad b = -\frac{1}{2}$

$\ker\varphi = \langle 6\mu - \lambda \rangle$

$\pi_1(T)/\ker\varphi = \langle \bar{\mu} \rangle$

cone angle $= 4\pi$

$\ell(\text{core}) = \operatorname{Re}(\tilde{g}_\mu)$.

- If a or $b \in \mathbb{R}$: Result is a cone mfld with cone angle $\operatorname{Im}(\tilde{g}_{\alpha_1})$, where α_1 generates $\ker\varphi$.
  - this is "generalized Dehn filling or surgery", but is not Dehn filling by our def-n.

- Our approach so far has been to start with a hyp-c structure on a triangulated mfld M, and understand how completion of the structure can be understood as Dehn filling.

Let's turn this around: start with a topological 3-mfld M w/ torus cusps and consider the (p,q)-Dehn filling of M. When does $M_{(p,q)}$ admit a complete hyp-c structure?

We have seen:

A: When edge eqns. and satisfied and the equation $p\tilde{g}_\mu + q\tilde{g}_\lambda = 2\pi i$ is a satisfied.

Can we say anything more general?

First, an extremely important theorem:

Theorem (Mostow–Prasad rigidity): If M_1 and M_2 are complete hyperbolic n-mflds with finite volume and $n \geq 3$, then any isomorphism $\varphi: \pi_1(M_1) \longrightarrow \pi_1(M_2)$ is realized by a unique isometry $g: M_1 \to M_2$ (i.e., $g_* = \varphi$).

Furthermore, letting $\Gamma_1 \cong \pi_1(M_1)$, $\Gamma_2 \cong \pi_1(M_2)$ be the holonomy groups, $\exists\, g \in \mathrm{Isom}(\mathbb{H}^3)$ such that

$$\varphi(\gamma) = g \circ \gamma \circ g^{-1} \text{ for all } \gamma \in \Gamma_1$$

(i.e., the isomorphism φ is realized by conjugation).

proof: see B+P, Ratcliffe '06

And on a related note:

Theorem (Gordon - Luecke): If K_1 and K_2 are knots with homeo-c complements, then the knots are isotopic, up to reflection.

Def-n: Let M be a 3-mfld with cusp torus T. The subset of $\mathbb{R}^2 \cup \{\infty\}$ consisting of Dehn filling coefficients of hyp-c structures on M is called the hyp-c Dehn filling space of M, where ∞ corresponds to the complete structure on M, if it exists.

Rigidity

Thm (Thurston's hyp-c Dehn filling thm): Let M be a 3-mfld with a single torus cusp T s.t. M admits a complete hyp-c structure. Then hyp-c Dehn filling space contains an open nbhd. of ∞ in $\mathbb{R}^2 \cup \{\infty\}$.

More generally, if M has cusps $T_1, \dots, T_k$, and M admits a complete hyp-c structure, then hyp-c Dehn filling space for M contains an open nbhd. of ∞ for each T_i.

(Thurston '79 (sketch), Neumann - Zagier '85, Benedetti - Petronio '92, Petronio - Porti '00, Hodgson - Kerckhoff '05 (effectivised)

Definition: A Dehn filling of a hyp-c mfld that does not admit a (complete) hyp-c metric is called exceptional.

Corollary: Let M be a mfld with a single torus cusp s.t. M admits a complete hyp-c structure. Then M has finitely many exceptional fillings (non-generalized).

Corollary: Let M be a mfld with torus cusps $T_1, \dots, T_k$. For each T_i, exclude finitely many fillings. The remaining Dehn fillings yield mflds with complete hyp-c structures.

Note: The second corollary excludes infinitely many fillings.

Consequence: "Most" Dehn fillings of hyp-c mflds are hyp-c.

- Wallace-Lickorish $\Rightarrow$ all closed mflds come from Dehn filling links, which may be taken to be hyp-c (Myers '93), so "most" closed 3-mflds are hyp-c.

Later we'll see: 6-theorem.

Also, for knots, ≤ 10 exceptional fillings.

6.4 Geometric Convergence:

Theorem 6.25: Let M admit a complete hyp-c structure, and fix a horoball neighborhood of a cusp C. Let s_n be a sequence of slopes on ∂C such that the length of (a geodesic rep-ve of) s_n, measured in the induced Euclidean metric on ∂C, approaches ∞.

Then for large enough n, the Dehn filled mflds $M(s_n)$ are hyperbolic and approach M as a geometric limit.

Goal: understand this theorem.

Roughly speaking, $M_n \to M$ as a geometric limit means that geometric invts. of M_n are close to those of M for large n.

E.g.: $M_n \to M \Rightarrow \mathrm{vol}(M_n) \longrightarrow \mathrm{vol}(M)$

In fact:

Thm 6.26 (Jörgensen's Thm): If $M(s)$ is obtained by Dehn filling M and both are hyp-c, then

$$\mathrm{vol}(M(s)) < \mathrm{vol}(M).$$

so $\mathrm{vol}(M_n)$ converges to $\mathrm{vol}(M)$ from below.

6.4.1 Convergence of spaces

(see Benedetti e Petronio, Canary et. al 2006, Cooper et al 2000).

Def-n 6.28: Let X and Y be metric spaces with distance function d_X and d_Y, resp. For $K>0$, a bijection $f: X \to Y$ is K-bilipschitz if for all $x, y \in X$,

$$\frac{1}{K} d_X(x,y) \leq d_Y(f(x), f(y)) \leq K\, d_X(x,y)$$

Def-n 6.29: Let X and Y be compact metric spaces. Define the bilipschitz distance to be

$$\inf \{ |\log \operatorname{bilip}(f)| + |\log \operatorname{bilip}(f^{-1})| \}$$

where the infimum is taken of all bilipshcitz maps from $X \to Y$, and $\operatorname{bilip}(f)$ denotes the bilipschitz constant, i.e., the minimal K for def-n 6.28. If there is no bilipschitz map from X to Z, then the bilipschitz distance is ∞.

- bilipschitz distance $\rightsquigarrow$ bilipschitz topology on the set of compact metric spaces.

Def-n 6.30: Let $\{X_n\}$ be a sequence of locally compact metric spaces with distinguished basepoint $x_n \in X_n$ for each n. The sequence $\{(X_n, x_n)\}$ is said to converge in the pointed bilipschitz topology to (X, x) if for any $R > 0$, the closed nbhds $B_R(x_n)$ of radius R about $x_n \in X_n$ converge to the closed neighborhood $B_R(y)$ about x in X in the bilipschitz topology.

Rmk: This allows us to talk about convergence of non-compact spaces (bilipschitz topology is for compact metric spaces).

We want something stronger:

Def-n 6.31: Let $\{X_n\}$ be a sequence of locally compact metric spaces with distinguished basepoint $x_n \in X_n$ and orthonormal basis v_n of $T_{x_n} X_n$, $\forall n$. The sequence $\{(X_n, x_n, v_n)\}$ converges in the framed pointed bilipschitz topology to (X, x, v) if for sufficiently large $R > 0$ and all $K > 1$, $\exists\ n_0$ s.t. for $n \geq n_0$, there are open nbhds. U_n of $B_R(x_n)$ and U of $B_R(y)$, and K-bilipschitz diffeo-sm $f_n : (U, v) \to (U_n, v_n)$ with $f_n(x) = x_n$ and $D_y f_n(v) = v_n$.

Also called: geometric convergence, & convergence in the refined Gromov-Hausdorff topology.

6.4.2 Convergence of discrete groups

If M is a hyperbolic 3-mfld, then $M \cong \mathbb{H}^3/\Gamma$ for some $\Gamma \subseteq PSL_2\mathbb{C}$ discrete, torsion free.

Thus, given a sequence $\{M_n = \mathbb{H}^3/\Gamma_n\}$ of hyp-c 3-mflds, we can consider the associated sequence $\{\Gamma_n\}$ of holonomy groups.

Def-n: Let G be a group (e.g., $\pi_1(M)$) and let $\rho_i : G \to PSL_2\mathbb{C}$ be a sequence of representations. $\{\rho_i\}$ converges algebraically to $\rho(G)$ if for every $\gamma \in G$, $\rho_i(\gamma) \to \rho(\gamma)$.

Def-n: A sequence of discrete groups $\Gamma_n \leq PSL_2\mathbb{C}$ converges geometrically to Γ_∞ if

(1) for any convergent sequence $\{\gamma_{n_j}\} \subseteq \Gamma_n$, $\lim \gamma_{n_j} \in \Gamma_\infty$

(2) for any $\gamma \in \Gamma_\infty$, there is a sequence $\gamma_n \in \Gamma_n$ s.t. $\lim \gamma_n = \gamma$.

- Also called convergence in the Chabauty topology.

Thm 6.34: TFAE:

(1) Discrete, torsion free groups $\Gamma_n \leq PSL_2\mathbb{C}$ converge geometricall to Γ_∞

(2) There exist basepoints $x_n \in \mathbb{H}^3/\Gamma_n$ and $x_\infty \in \mathbb{H}^3/\Gamma_\infty$, and oriented frames v_n and v_∞ for $T_{x_n}(\mathbb{H}^3/\Gamma_n)$ and $T_{x_\infty}(\mathbb{H}^3/\Gamma_\infty)$ s.t. $(\mathbb{H}^3/\Gamma_n, x_n, v_n)$ converges to $(\mathbb{H}^3/\Gamma_\infty, x_\infty, v_\infty)$ in the framed pointed bilipschitz topology.

Example:

Let M be a hyperbolic 3-mfld with a single torus cusp, such that the $(1,-n)$ Dehn filling $M_{(1,-n)}$ is hyp-c for $n \geq 9$ (w.r.t. a basis μ, λ for $\pi_1(T)$)

Let Γ_n be the holonomy group for $M_{(1,-n)}$, so that $\rho_n : \pi_1(M) \longrightarrow \Gamma_n \subseteq PSL_2\mathbb{C}$.

since rotation angle of μ and λ are both $< 2\pi$, we can just use holonomies.

Then $\quad 2\pi i = \tilde{g}_\mu - n\tilde{g}_\lambda = g_\mu g_\lambda^{-n}$

$$= \rho_n(\mu)\, \rho_n(\lambda)^{-n}$$

by def'n of g_μ, g_λ

Since a rotation by $2\pi i$ is the identity in $PSL_2\mathbb{C}$,

$$\rho_n(\mu) = \rho_n(\lambda)^n.$$

Consider $\langle \rho_n(\mu), \rho_n(\lambda) \rangle = \langle \rho_n(\mu) \rangle \cong \mathbb{Z}$

as rep-ns $\varphi_n : \mathbb{Z} \longrightarrow PSL_2\mathbb{C}.$, $\varphi_n(1) = \rho_n(\mu)$.

As reps, φ_n converge to a parabolic rep-n of $\mathbb{Z}$ into $PSL_2\mathbb{C}$

But $\varphi_n(\mathbb{Z}) = \langle \rho_n(\mu) \rangle$ converges to a rank 2 parabolic subgroup $\mathbb{Z} \times \mathbb{Z}$ gen. by $\lim \rho_n(\mu)$ and $\lim \rho_n(\lambda)^n$

Ch. 5: Discrete Groups & Thick-thin decomposition

Main Goal: Decomposition of hyp-c 3-mflds into 'thick' part and simple 'thin' parts

5.1 Discrete subgroups of $Isom^+(\mathbb{H}^3) = PSL_2\mathbb{C}$.

Def'n: A subgroup $\Gamma \le PSL_2\mathbb{C}$ is discrete (or, Kleinian) if it contains no sequence of distinct elts. converging to the identity.

Equivalently,

Lemma 5.5: A subgroup $\Gamma \le PSL_2\mathbb{C}$ is discrete $\Leftrightarrow$ it contains no sequence of distinct eltr. converging to an elt. $A \in PSL_2\mathbb{C}$

proof: easy $\left(A_n \longrightarrow A \quad \Rightarrow \quad A_n A^{-1} \longrightarrow Id\right)$

(Purcell does $A_{n+1}A_n^{-1} \longrightarrow Id$. Why?)

- In general, finding discrete subgroups of $PSL_2\mathbb{C}$ is hard.

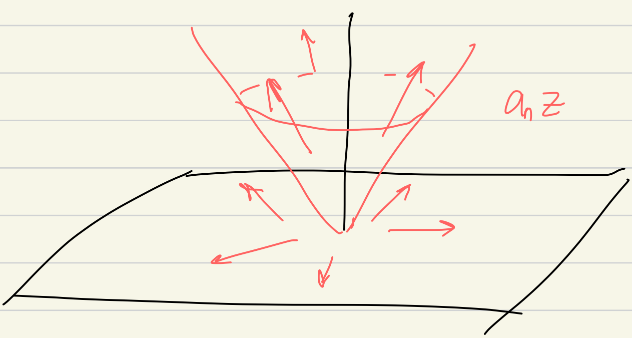
Goal: Show that holonomy group of complete hyp-c structures are discrete.

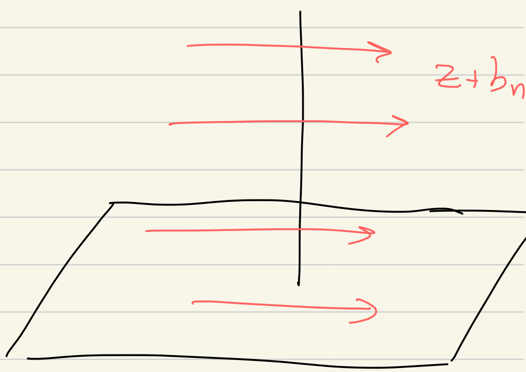
Lemma 5.6: Let $\{A_n\}$ be a sequence of elts. of $PSL_2\mathbb{C}$. Then either a subsequence of $\{A_n\}$ converges to some $A \in PSL_2\mathbb{C}$, or $\exists\ q \in \partial\mathbb{H}^3$ s.t. for all $x \in \mathbb{H}^3$, $\{A_n(x)\}$ has a subsequence converging to q.

proof: idea: look at fixed pts. $p_n, q_n \in \partial\mathbb{H}^3$ of A_n. $\partial\mathbb{H}^3$ compact $\Rightarrow$ conv. subsequences to p, q. After conjugation, may assume $p=0$, $q=\infty$, or $p=q=\infty$. w.l.o.g. may assume in both cases that $q=\infty$ is attracting fixed pt. So $A_n = a_n z$ or

$A_n = b_n z + a_n$, with $b_n \to 1$. If $|a_n|$ is bounded,

$a_n \to a$, so $A_n \to A$, $A = az$ or $z+a$.
If a_n unbounded, $A_n(x) \to q$ for all $x \in \mathbb{H}^3$, since q is attracting fixed pt.





Details: Purcell. $\square$

Def-n: $\Gamma \leq PSL_2\mathbb{C}$.

- $\Gamma \curvearrowright \mathbb{H}^3$ is <u>properly discontinuous</u> if for every close ball $B \subseteq \mathbb{H}^3$, $\{\gamma \in \Gamma \mid \gamma(B) \cap B \neq \emptyset\}$ is finite.

- $\Gamma \curvearrowright \mathbb{H}^3$ is <u>free</u> if only $Id \in \Gamma$ has a fixed point.

($\Gamma \curvearrowright$ is free $\iff$ Γ contains no elliptics)

Lemma: $\Gamma \leq PSL_2\mathbb{C}$ is discrete $\iff$ $\Gamma \curvearrowright \mathbb{H}^3$ is properly discontinuous.

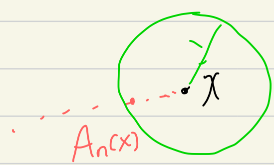
proof: ($\Leftarrow$) suppose G is not discrete, so $\exists\ A_n \to Id$.

$\therefore \forall x \in \mathbb{H}^3$, $d(x, A_n(x)) \xrightarrow{n\to\infty} 0$

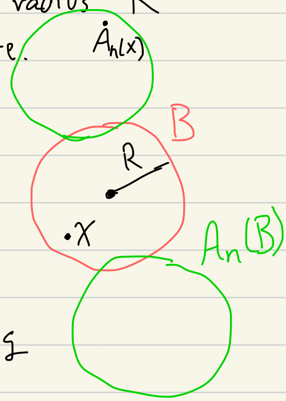
Let B be a closed ball about x of radius 1.
Then $d(x, A_n(x)) < 1 \Rightarrow$

$$A_n \in \{A \in \Gamma \mid A(B) \cap B \neq \emptyset\},$$

so the set is infinite.



($\Rightarrow$) suppose $\exists$ a closed ball B of radius R s.t. $S = \{A \in \Gamma \mid A(B) \cap B \neq \emptyset\}$ in infinite.



Let $\{A_n\} \subseteq S$ be distinct.

Note that $d(x, A_n(x)) \leq 4R$, $\forall n$.

$\therefore$ $A_n(x)$ has no subsequence converging to a point in $\partial \mathbb{H}^3$.

By Lemma 5.6, A_n has a subsequence converging to $A \in PSL_2\mathbb{C}$.

$\therefore$ Γ not discrete by Lemma 5.5. $\square$

<u>Prop-n</u>: $\Gamma \curvearrowright \mathbb{H}^3$ is free and properly disc.

$\Leftrightarrow$ $\mathbb{H}^3/\Gamma$ is a hyp-c 3-mfld. with covering projection $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma$.

proof: ($\Rightarrow$) Suppose $\Gamma \curvearrowright \mathbb{H}^3$ is free + P.D.

Let $x \in \mathbb{H}^3/\Gamma$, $\tilde{x} \in p^{-1}(x)$, where $p: \underset{\Gamma \curvearrowright \mathbb{H}^3}{\mathbb{H}^3} \to \mathbb{H}^3/\Gamma$.

Since action is free + P.D., there is a closed ball $B_{\tilde{x}}$ about $\tilde{x}$ s.t. $B_{\tilde{x}}$ is disjoint from all its translates by Γ.

$\therefore$ $\mathrm{int}(B_{\tilde{x}})$ maps isometrically to a nbhd. of x via p, so $\mathbb{H}^3/\Gamma$ is a hyp-c mfld. Since this nbhd. is evenly covered by the translates of $B_{\tilde{x}}$, p is a covering projection.

($\Leftarrow$) Now suppose $\mathbb{H}^3/\Gamma$ is hyp-c and $p: \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ is a covering projection.

For any $x \in \mathbb{H}^3$, Γ permutes $p^{-1}(p(x))$, and only $Id \in \Gamma$ fixes x, so the action is free.

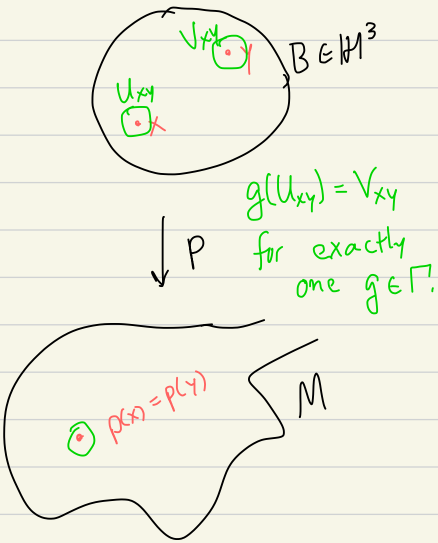
Let $B \subseteq \mathbb{H}^3$ be a closed ball.

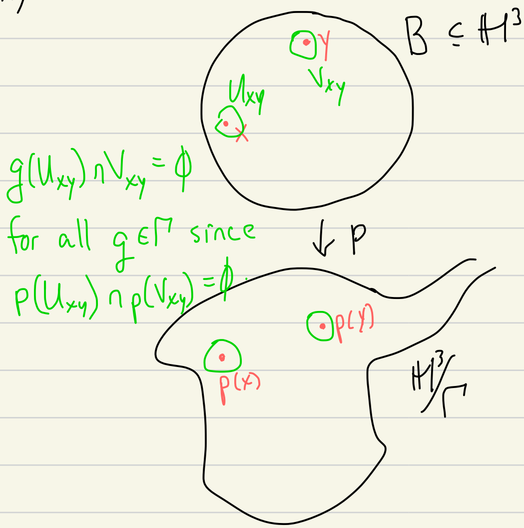
Claim: for any $x, y \in B$, $\exists$ nbhds $U_{xy} \ni x$ and $V_{xy} \ni y$ s.t. $g(U_{xy}) \cap V_{xy} \neq \emptyset$ for at most one $g \in \Gamma$.

non-trivial.

proof of Claim: If $p(x) \neq p(y)$, take disjoint nbhds of $p(x)$ and $p(y)$ and lift to $\mathbb{H}^3$ to get U_{xy} and V_{xy}.

If $p(x) = p(y)$, take a nbhd. of $p(x)$ that is evenly covered and lift to $\mathbb{H}^3$ to get U_{xy}, V_{xy}.





- $(x, y) \in B \times B$, and the sets $U_{xy} \times V_{xy}$ cover B, so $\exists$ finite subcover

$$\{U_1 \times V_1, \ldots, U_n \times V_n\} \text{ with associated elts.}$$

$g_1, \ldots, g_n$ as described above (some maybe $= Id$).

If $\gamma \in \Gamma$ and $x \in \gamma(B) \cap B$, then $(\gamma^{-1}(x), x) \in B \times B$.

$\therefore$ $(\gamma^{-1}(x), x) \in U_i \times V_i$ for some i, so $\gamma = g_i$

$\therefore$ $\gamma \in \{g_1, \ldots, g_n\}$, so the action is P.D. $\square$

5.2: Elementary groups

Def-n: A subgroup $\Gamma \leq PSL_2\mathbb{C}$ is elementary if either $\exists$ $x \in \mathbb{H}^3$ s.t. $\Gamma_x = \Gamma$, or
$\uparrow$ stabilizer of x.

$$Fix(\Gamma) = \{ x \in \partial\mathbb{H}^3 \mid \gamma(x) = x \text{ for some } Id \neq \gamma \in \Gamma \}$$

contains at most two elts. (i.e., one or two).

- Else: non-elementary.

Prop-n 5.12: If $\Gamma \leq PSL_2\mathbb{C}$ is discrete and torsion free (i.e., contains no elliptics), and is a nontrivial elementary group, then either

(1) $|Fix(\Gamma)| = 1$ and $\Gamma \cong \mathbb{Z}$ or $\Gamma \cong \mathbb{Z} \times \mathbb{Z}$, and Γ is generated by parabolics.

(2) $|Fix(\Gamma)| = 2$, $\Gamma \cong \mathbb{Z}$, and Γ is gen. by a loxodromic with axis connecting the fixed pt.

proof: exercise.

Corollary: If $\Gamma \le PSL_2\mathbb{C}$ is discrete, then any $\mathbb{Z}\times\mathbb{Z}$ subgroup is generated by a pair of parabolics with a common fixed pt.

Def-n: Let $\Gamma \le PSL_2\mathbb{C}$ be a torsion free Kleinian group, and let $\Gamma_\infty \le \Gamma$ be a non-trivial elementary subgroup with a single fixed pt, which we may assume (by conjugating Γ) is ∞.

Let H be the horoball of height 1 about ∞:

$$H = \{(x, y, z) \mid z \ge 1\}$$

If $\Gamma_\infty \cong \mathbb{Z}$, then $H/\Gamma_\infty \cong A \times [1,\infty)$,

where A is an annulus. We say H/Γ_∞ is a rank-1 cusp.

If $\Gamma_\infty \cong \mathbb{Z}\times\mathbb{Z}$, then $H/\Gamma_\infty \cong T^2 \times [1,\infty)$,

where T^2 is a Euclidean torus. We say H/Γ_∞ is a rank-2 cusp.

<u>Lemma 5.18</u>: If Γ is a non-elementary discrete torsion free subgroup of $PSL_2\mathbb{C}$, then

(1) $|\Gamma| = \infty$

(2) For any nontrivial $A \in \Gamma$, $\exists$ a loxodromic $B \in \Gamma$ that shares no fixed point with A.

(3) If $B \in \Gamma$ is loxodromic, then no nontrivial $C \in \Gamma$ has exactly one fixed point in common with B.

(4) Γ contains two loxodromics with <u>no</u> fixed pts. in common.

proof: (1): Γ non-trivial, torsion-free $\Rightarrow$ contains an infinite order elt. $\Rightarrow$ (1)

(3): Suppose $B \in \Gamma$ is loxodromic, and $\exists\ C \in \Gamma$ having a fixed pt. in common with B.

Conjugation $\rightsquigarrow$ may assume $\underbrace{B = \begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix}}_{\text{fixed pts. } 0 \text{ and } \infty}$, $|\rho| > 1$,

$C = \underbrace{\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}}_{\text{fixed pt. } \infty}$. Then

$$B^n C B^{-n} C^{-1} = \begin{pmatrix} 1 & ab(\rho^{2n} - 1) \\ 0 & 1 \end{pmatrix}$$

Let $n \to -\infty$. Then

$$B^n C B^{-n} C^{-1} \longrightarrow \begin{pmatrix} 1 & -ab \\ 0 & 1 \end{pmatrix}$$

Lemma 5.5 $\Rightarrow \langle B, C \rangle$ not discrete

$\Rightarrow \Gamma$ not discrete.

Note: we only proved conj. to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, but this is true.

(2) Case 1: A is parabolic.

Conjugation of $\Gamma \rightsquigarrow A = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\text{fixes } \infty}$

Γ non-elementary $\Rightarrow \exists\ C \in \Gamma$ that does not fix ∞. If C is loxodromic, we're done. otherwise

since C does not fix ∞.

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad c \neq 0, \quad \text{and} \quad AC = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

has trace $\operatorname{tr}(AC) = a+c+d = c \pm 2 \notin \{-2, 2\}$.

$\therefore AC$ is not parabolic, must be loxodromic.

Γ is torsion-free, so no elliptics

Case 2: A is loxodromic.

fixes 0 and ∞

conjugation of $\Gamma \rightsquigarrow A = \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}$, $|p|>1$.

Γ non-elementary $\Rightarrow \exists\, C \in \Gamma$ s.t. the fixed points of C are not 0 and ∞. By (3), C cannot have exactly one fixed pt. in common with A, so C must have no fixed pts. in common with A.

If C is loxodromic, we are done. Otherwise,

$$A^nC = \begin{pmatrix} ap^n & bp^n \\ cp^{-n} & dp^{-n} \end{pmatrix} \quad \text{has trace}$$

$tr(A) = ap^n + dp^{-n}$, which is not $= \pm 2$ for n large. Also, A^nC does not fix 0 or ∞.

($C: 0 \mapsto z \neq 0$, $A^n: z \mapsto w \neq 0$, since C does not fix 0, and $A^n(0)=0$.

(4): Immediate from (2). $\square$

We will also need: of a Thm. of Jørgensen & Klein.

Corollary 5.20: Suppose $\{\langle A_n, B_n\rangle\}$ is a sequence of non-elementary discrete subgroups of $PSL_2\mathbb{C}$ s.t. $\lim A_n = A$ and $\lim B_n = B$ in $PSL_2\mathbb{C}$. Then $\langle A, B\rangle$ is a non-elementary discrete subgroup of $PSL_2\mathbb{C}$.

proof: (see Marden '07)

5.3 Univ. Elementary Neighborhoods.

Def-n: • Suppose M is a complete hyp-c 3-mfld, $x \in M$. The injectivity radius of x, denoted $\mathrm{injrad}(x)$, is the supremal radius r such that a metric r-ball is embedded.

• Let M be a complete hyp-c 3-mfld and let $\varepsilon > 0$. Define the ε-thin part of M to be

$$M^{<\varepsilon} = \{ x \in M \mid \mathrm{injrad}(x) < \varepsilon/2 \}$$

• Define the ε-thick part to be

$$M^{>\varepsilon} = \{ x \in M \mid \mathrm{injrad}(x) > \varepsilon/2 \}$$

• Also, $M^{\leq \varepsilon}$ and $M^{\geq \varepsilon}$ are defined analogously.

Theorem 5.23: There exists a universal constant $\varepsilon_3 > 0$ such that for $0 < \varepsilon \leq \varepsilon_3$, the ε-thin part of any complete, orientable, hyp-c 3-mfld M consists of tubes around short geodesics, and rank-1 and rank-2 cusps.

• the supremum of all constants satisfying Thm 5.23 is called the Margulis constant.

Rmk: Known bounds on the Margulis number:

$$0.104 \le \varepsilon_3 \le 0.616$$

Meyerhoff '87 ↗ ↖ Culler

[Každen – Margulis '68]

Thm 5.23 is a consequence of the Margulis Lemma, which is a more general version of the following:

Due to Jørgensen + Marden (see [Marden '07])

holds without this condition (see [Wang '67])

$\boxed{\text{Thm 5.25}}$ (Universal elementary nbhds): There is a universal constant $\varepsilon_3 > 0$ such that for all $x \in \mathbb{H}^3$, and for any discrete $\boxed{\text{torsion-free}}$ group $\Gamma \le PSL_2\mathbb{C}$, the subgroup $H \le \Gamma$ generated by elts. of Γ that translate x distance less that ε_3 is elementary

$$H = \langle \{ \gamma \in \Gamma \mid d(x, \gamma(x)) < \varepsilon_3 \} \rangle$$

proof: For $x \in \mathbb{H}^3$, $\Gamma \le PSL_2\mathbb{C}$, and $r > 0$, let

$$\Gamma(r, x) = \{ \gamma \in \Gamma \mid d(x, \gamma(x)) < r \}$$

and let $\langle \Gamma(r,x) \rangle$ be the group generated by $\Gamma(r,x)$.

We need to show that $\exists\, r>0$ s.t. for any $x \in \mathbb{H}^3$ and discrete, torsion-free $\Gamma \leq PSL_2\mathbb{C}$, $\langle \Gamma(r,x) \rangle$ is elementary.

First we show that for fixed Γ and x, there exists $r>0$ s.t. $\langle \Gamma(r,x) \rangle$ is elem.

If such an r did not exist, then we could find a sequence $r_n \to 0$ s.t. $\langle \Gamma(r_n, x) \rangle$ is non-elem. for each r_n.

$\therefore$ we can find distinct $A_n \in \Gamma$ s.t. $d(x, A_n(x)) < r_n$.

Lemma 5.6 $\Rightarrow$ $A_n \longrightarrow A \in PSL_2\mathbb{C}$

$\Rightarrow \Gamma$ not discrete.

$\therefore$ for small enough r, $\langle \Gamma(r,x) \rangle$ is elementary.

It follows that $\Gamma(r,x)$ is finite, so we can take r ever smaller so that $\Gamma(r,x) = \{Id\}$ (just take r smaller than the fin. many translation distances for elts. in $\Gamma(r,x)$)

Upshot: For fixed Γ, x, $\exists\, r>0$ s.t.

$\langle \Gamma(r,x) \rangle = \{Id\}$ (which is elementary).

We now show that there is a universal constant $r>0$ s.t. for any choice of Γ, x, the group $\langle \Gamma(r,x) \rangle$ is elementary.

If not, then we can find sequences $\{\Gamma_n\}$, $\{x_n\}$, $\{r_n\}$ so that $\langle \Gamma_n(r_n, x_n) \rangle$ is non-elem.

To simplify things, fix some $x \in \mathbb{H}^3$, and let $R_n \in PSL_2\mathbb{C}$ be s.t. $R_n(x_n) = x$.

Then $\langle \Gamma_n(r_n, x_n) \rangle$ is non-elem.

$$\Rightarrow \langle R_n \Gamma_n R_n^{-1}(r_n, x) \rangle \text{ is non-elem.}$$

$\therefore$ we may assume that $x_n = x \ \forall n$
(we will abuse notation and still use Γ_n for $R_n \Gamma_n R_n^{-1}$).

Now fix n. We will find $A_n, B_n \in \Gamma_n(r_n, x)$ s.t. $\langle A_n, B_n \rangle$ is non-elem.

Since $\langle \Gamma_n(r_n, x) \rangle$ is non-elem., $\Gamma_n(r_n, x)$ contains at least 2 elements.

If $\langle A_n, B_n \rangle$ is elementary for <u>every</u> choice of A_n, B_n in $\Gamma_n(r_n, x)$, then for every such pair: (1) A_n, B_n are loxodromics translating along a common axis, with $A_n = (B_n)^k$, or

(2) A_n, B_n are parabolics with a common fixed pt.

It follows that $\Gamma_n(r_n, x)$ must consist either of loxodromics all with a common axis, or parabolics w/ a common fixed pt.

In the first case, all the loxodromics are generated by a single elt., otherwise Γ_n would not be discrete.

$\therefore$ In both cases, $\langle \Gamma_n(r_n, x) \rangle$ is elementary, a contradiction.

$\therefore$ for each n, $\exists\ A_n, B_n \in \Gamma_n(r_n, x)$ s.t.

$\langle A_n, B_n \rangle$ is non-elementary.

Since $r_n \to 0$, $A_n(x) \to x$ and $B_n(x) \to x$.

$\therefore$ by Lemma 5.6, $A_n \to A$ and $B_n \to B$,

$A, B \in PSL_2\mathbb{C}$.

$\therefore$ by Corollary 5.20 $\langle A, B \rangle$ is non-elementary.

On the other hand, $A(x) = x$ and $B(x) = x$, so

$\langle A, B \rangle$ is elementary $(\mathrm{Stab}(x) = \langle A, B \rangle)$.

$\square$

Recall: $\mathrm{injrad}(x) = \sup\{r \mid B(r,x) \text{ is embedded}\}$

To relate translation distance (as in Thm 5.25) to injectivity radius (as in Thm 5.23), we need:

<u>Lemma 5.26</u>: Let M be a complete, orientable, hyperbolic 3-mfld with $M = \mathbb{H}^3/\Gamma$ for a discrete group $\Gamma \leq PSL_2\mathbb{C}$.

For any $x \in M$ with lift $\tilde{x} \in \mathbb{H}^3$,

$$\mathrm{injrad}(x) = \frac{1}{2} \inf_{\mathbb{1} \neq A \in \Gamma} \{d(\tilde{x}, A\tilde{x})\},$$

and this inf is realized by some non-trivial $A \in \Gamma$.

<u>proof</u>: An r-ball is embedded at x if and only if for all $Id \neq A \in \Gamma$, the r-ball

$B(r,\tilde{x})$ is disjoint from the r-ball

$A(B(r,\tilde{x})) = B(r, A\tilde{x})$. This holds if and only if $d(\tilde{x}, A\tilde{x}) \geq 2r$ for all A.

Now, suppose $\mathrm{injrad}(x) = b$. Then $B(b,x)$ is embedded, but $B(b+\varepsilon, x)$ is not for $\varepsilon > 0$.

$\therefore\ \forall\ 0 < \varepsilon < 1$, $\exists\ A_\varepsilon \in \Gamma$ s.t. $d(\tilde{x}, A_\varepsilon(\tilde{x})) < 2(b+\varepsilon)$

If the set $\{A_\varepsilon\}$ contains infinitely many distinct elts., we get a sequence $\{A_n\}$ s.t. $d(\tilde{x}, A_n(\tilde{x}))$ is bounded

$\therefore$ by Lemma 5.6 $A_n \longrightarrow A \in PSL_2\mathbb{C}$,

so Γ is not discrete, a contradiction.

$\therefore \{A_\varepsilon\}$ is finite

since we can get as close to b as we want, but there are finitely many A_ε to do this with.

It follows that for some $A \in \{A_\varepsilon\}$,

$d(\tilde{x}, A(\tilde{x})) = b$.

$\square$

proof of Thm 5.23:

Take $\varepsilon_3 > 0$ as provided by Thm. 5.25.

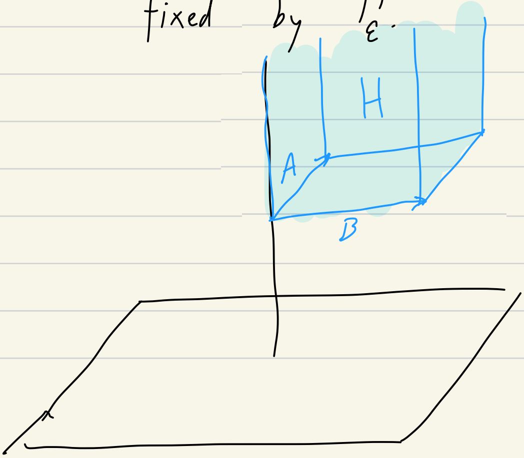
Let $M = \mathbb{H}^3/\Gamma$ be complete, orientable, hyp-c 3-mfld, so Γ is discrete, torsion-free.

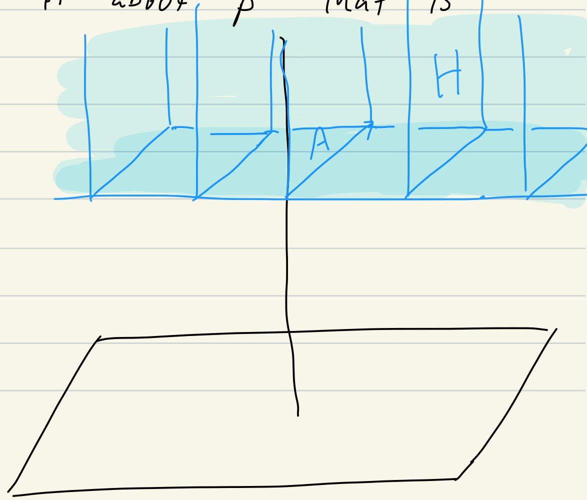
For $\varepsilon < \varepsilon_3$, if $x \in M^{<\varepsilon}$ then by def-n $\text{injrad}(x) < \frac{\varepsilon}{2}$

$\therefore$ by Lemma 5.26 $\exists\ \Gamma \ni A \neq Id$ s.t. $d(\tilde{x}, A\tilde{x}) < \varepsilon$ for any lift $\tilde{x}$ of x.

Thm 5.25 $\Longrightarrow$ the subgroup Γ_ε generated by such A is elementary, so Γ_ε is generated by a single loxodromic, or (one or two) parabolics with a common fixed pt. $p \in \partial\mathbb{H}^3$

Case 1: Γ_ε fixes a single pt. $p \in \partial\mathbb{H}^3$. Then $\tilde{x}$ lies on a horosphere H about p that is fixed by Γ_ε.





for any $\tilde{y} \in H$, $\varepsilon > d(\tilde{x}, A\tilde{x}) > d(\tilde{y}, A\tilde{y})$
for any generator A of Γ_ε.

$\therefore$ $\tilde{y}$ projects to $M^{<\varepsilon}$.

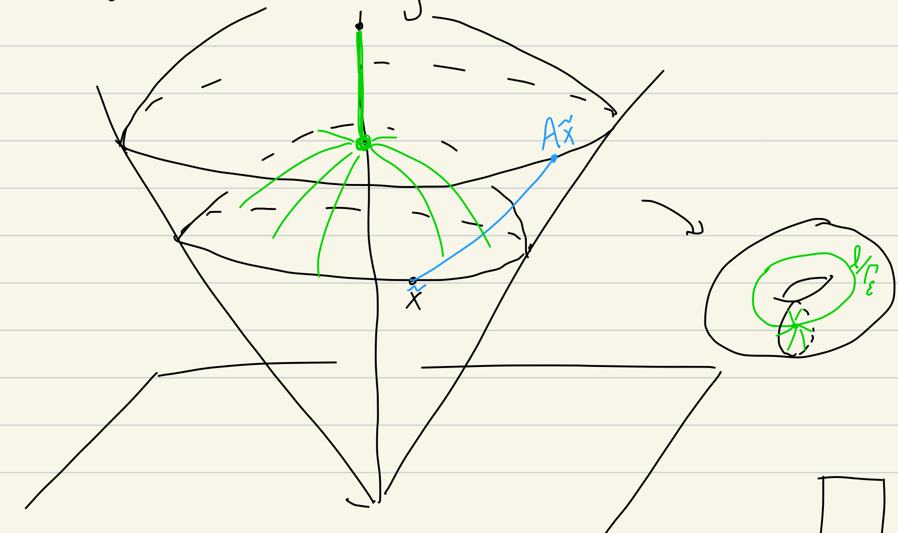
$\therefore$ $M^{<\varepsilon}$ contains the quotient of H, which is a rank-1 or rank-2 cusp.

Case 2: Γ_ε is generated by a loxodromic A with axis ℓ.

Let $R = d(\tilde{x}, \ell)$, $T_R = \{ \tilde{y} \in \mathbb{H}^3 \mid d(\tilde{y}, \ell) \leq R \}$

If $\tilde{y} \in T_R$ then $d(\tilde{y}, A\tilde{y}) < d(\tilde{x}, A\tilde{x}) < \varepsilon$,

so $M^{<\varepsilon}$ contains the quotient of T_R, which is a tube around a geodesic.



$\square$

5.4: Hyperbolic 3-mflds of finite volume

As an application of Thm. 5.23, we have:

Thm. 5.27. A hyperbolic 3-mfld M has finite volume $\iff$ M is closed, or M is homeomorphic to the interior of a compact mfld $\overline{M}$ with torus boundary components, where $\overline{M} \not\cong T^2 \times [0,1]$.

proof: ($\Leftarrow$) If M is closed, then a fund. domain for M is a compact set in $\mathbb{H}^3$, hence finite volume.

If $M = \operatorname{int}(\overline{M})$, where $\overline{M}$ has torus bdy, then $M \setminus \{\text{cusps}\}$ is compact (hence finite volume), and each cusp C of M has finite volume:

After an isometry of $\mathbb{H}^3$, we may assume that C lifts to a horoball H about ∞, with $\partial C = \partial H = \{(x,y,t) \in \mathbb{H}^3 \mid t = 1\}$.

Then $\operatorname{vol}(C) = \displaystyle\int_{t=1}^{\infty}\int_{\partial C} d\mathrm{vol} = \int_{t=1}^{\infty}\int_{\partial C} \frac{dx\,dy\,dt}{t^3}$

$$= \int_{\partial C}\left(\int_{t=1}^{\infty} \frac{1}{t^3}\,dt\right) dx\,dy = \frac{1}{2}\,\operatorname{area}(\partial C)$$

which is finite since ∂C is compact.

($\Rightarrow$) Suppose M has finite volume. Fix $0 < \varepsilon < \varepsilon_3$. By Thm 5.23 $M^{<\varepsilon}$ consists of tubes and cusps. Since rank-1 cusps have infinite volume, all cusps must be rank-2.

Since M is assumed to have finite volume, $M^{\geq \varepsilon}$ also must have finite volume.

Claim: $M^{\geq \varepsilon}$ is compact.

By def-n of $M^{\geq\varepsilon}$, any point $x \in M^{\geq \varepsilon}$ is contained in an embedded (in M) $\varepsilon/2$-ball $B_x(\varepsilon/2)$ about x. If $x, y \in M^{\geq\varepsilon}$ and $d(x,y) \geq \varepsilon$, then $B_x(\varepsilon/2)$ and $B_y(\varepsilon/2)$ are disjoint.

$\therefore$ A collection of points in $M^{\geq\varepsilon}$ s.t. any two points are at least distance ε from each other, gives a collection of disjoint $\varepsilon/2$-balls.

Since M has finite volume, any such collection is finite, and can be completed to a maximal collection $\{x_i\}$.

Since the collection is maximal, the closed ε-balls $\{\overline{B_{x_i}(\varepsilon)}\}$ cover $M^{\geq\varepsilon}$. The union of thes balls is compact in M, and $M^{\geq\varepsilon}$ is then compact as a closed subset of a compact set.

Let N be the union of $M^{\geq \varepsilon}$ and tubes in $M^{< \varepsilon}$. N is compact as a union of compact sets, and has torus boundary.

Since each cusp is homeo-c to $T^2 \times [0,1)$, If we attach a copy of $T^2 \times [0,1]$ to each bdy. component of N to get $\overline{M}$, then $M \cong \operatorname{int}(\overline{M})$.

$\square$

<u>Corollary</u>: Complements of hyperbolic knots and links in S^3 have finite volume.

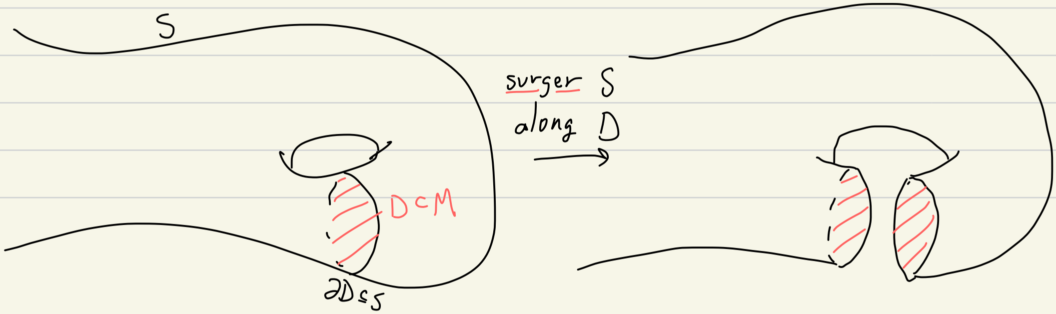
Chapter 8 : Essential surfaces

Rmk: All topological mflds in this section will be assumed to be smooth. Consequently:

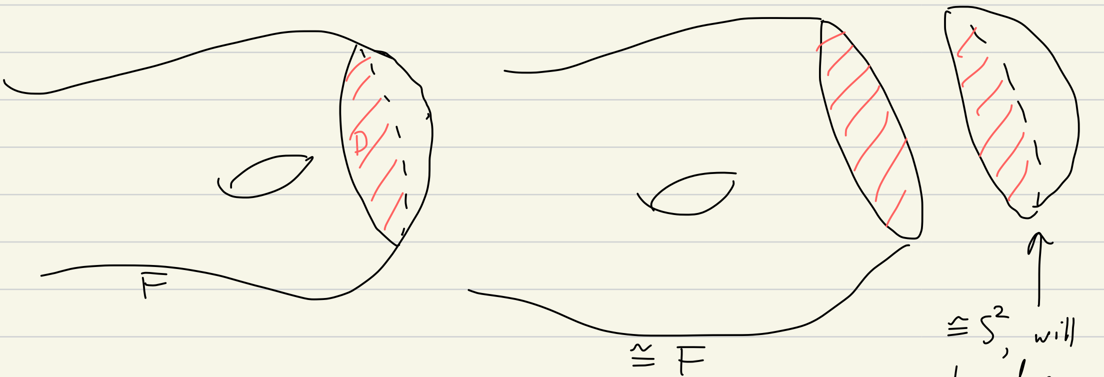
- submanifolds have tubular nbhds.
- isotopies of submanifolds can be extended to ambient isotopies
- Submanifolds can be perturbed to intersect transversely.

Also, all 3-mflds will be orientable, and surfaces will be typically be properly embedded.

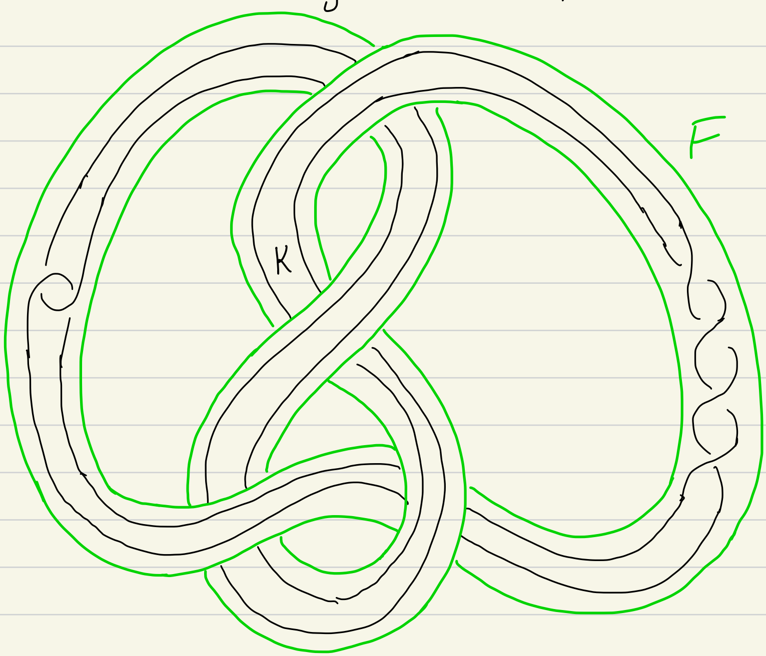
Def-n: Let $F \subset M^3$ be connected (and properly embedded). An embedded disk $D \subset M$ with $\partial D \subset F$ is a compressing disk for F if ∂D does not bound a disk in F.
A surface that admits a compressing disk is compressible. A surface that does not admit a compressing disk, and is not S^2, P^2, or D^2 is incompressible.
$\hookrightarrow$ 2-sphere $\hookrightarrow$ proj. plane $\hookrightarrow$ disk



If $D \subset M$ is isotopic into F (so not a compression disk), then surgering along D does not change S:

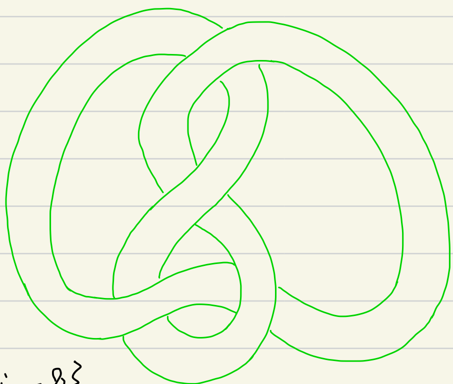


<u>Example</u>: Let K be the knot shown below (in black), and let $M \cong S^3 \setminus K$. Let F be the green torus.



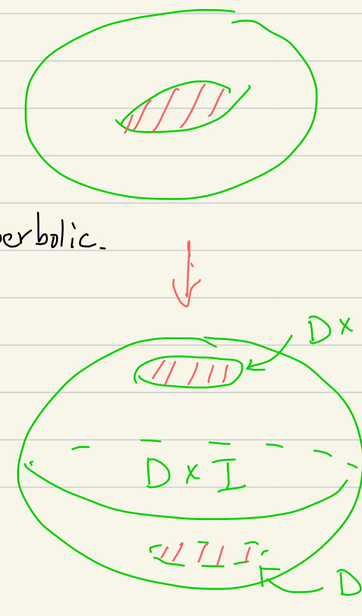
Note that F cuts M into an outer component homeo-c to $S^3 \setminus \{\text{fig-8}\}$, and in inner component homeo-c to $S^3 \setminus$  ← Whitehead link

Claim: the outside component does not contain a compression disk.

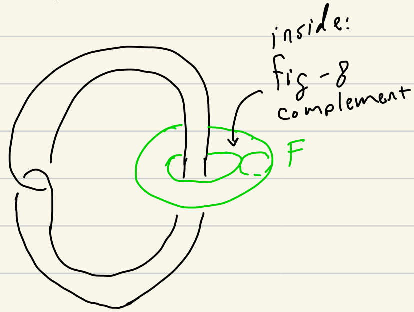


- Suppose such a disk D exists. F surgered along D is a sphere S. As a sphere in S^3, S bounds two balls. One of these contains K and D, and is naturally identified with $D \times I$. The other is a ball in $S^3 \setminus \{\text{fig-8}\}$. Reversing the surgery glues this ball to itself along two disks $D \times \{0\}$ and $D \times \{1\}$ $\subseteq S$.

  $\therefore$ $S^3 \setminus \{\text{fig-8}\}$ is a solid torus, the unknot complement, which is not hyperbolic. Contradiction.



If D is in the inner component, then F surgered along D is a sphere S, where separates S^3 into two balls, each containing one component of the Whitehead link.



This is impossible since the Whitehead link is not split.
(If it were, then would be $\cong S^3 \setminus \{ O \cup O \}$, so fund. grp. would be F_2, which it is not!)

$\therefore$ F is incompressible.

Definition: An embedded surface $F \subseteq M^3$ is boundary parallel if it can be isotoped into ∂M.
↳ or into a cusp if M has cusps.

Definition: A satellite knot is a knot that contains an incompressible torus that is not boundary parallel.

↳ Remove a knot K from a solid torus V, with K not contained in a ball or isotopic to the core of V, then "tie $V \setminus K$ in a knot" K', K' non-trivial.

<u>Lemma 8.7</u>: (1) An orientabe surface in a orientable 3-mfld is incompressible $\iff$ it is π_1-injective, i.e., $\pi_1(S) \xhookrightarrow{i_*} \pi_1(M)$.

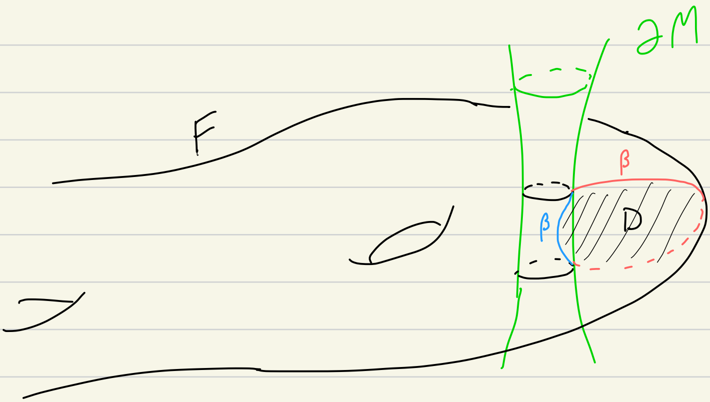
(2) A non-orientable surface S is π_1-injective $\iff$ $\partial(\underbrace{N(S)}_{\text{tubular nbhd. of } S})$ is orientable and incompressible.

Proving 8.7 requires the loop Theorem, which is classical

<u>Thm 8.48</u> (Papakyriakopoulos '57): If N is a 3-mfld with boundary, and there is a map $f: D^2 \to N$ such that the loop $f(\partial D^2) \subseteq \partial N$ is homotopically non-trivial in ∂N, then there is an embedding with the same property.

<u>proof of 8.7</u>: Let $S \subseteq M^3$ be orientable.
If $1 \neq \gamma \in \pi_1(S)$ and $i_*(\gamma) = 1$, then γ bounds a (possible immersed) disk in $N = M \setminus S$, with $\partial D \subseteq \partial N$. Since $\partial D = \gamma$ is homotopically nontrivial in S, we can realize an embedded such D, which is then a compression disk.
This proves $(\Longrightarrow)$.
If S is π_1-injective, then any non-trivial loop on S is non-trivial in M, so it cannot bound a compression disk.
(2) is left as a exercise. $\square$

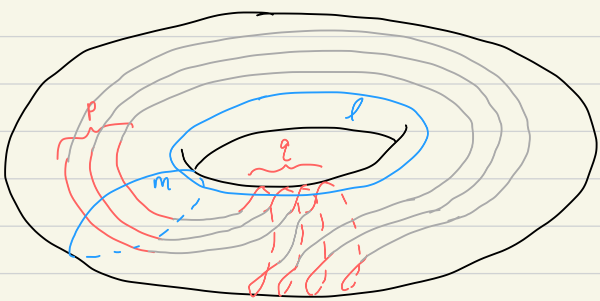
Definition: Let $F \subset M^3$, $\partial F \subset \partial M$. A boundary compression disk for F is a disk D with ∂D consisting of two arcs, $\partial D = \alpha \cup \beta$, s.t. $\alpha \subseteq F$ and $\beta \subseteq \partial M$, and such that there is no arc $\gamma \subset \partial F$ s.t. $\alpha \cup \gamma$ bound a disk in F.



If F admits a boundary compression disk, then we say it is boundary compressible, otherwise it is boundary incompressible.

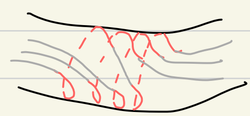
Definition: (p,q)-torus knot: $T(p,q) = K$

$|K \cap m| = |p|$
$|K \cap \ell| = |q|$



$(4,3)$-torus knot

- for negative p, twist in the other direction:



$(-4,3)$-torus knot

Example 8.10: Suppose $T(p,q) = K$ is a torus knot with $|p|, |q| \geq 2$. $A = T \setminus K$ is a annulus.
↖ the torus K lies on.

A is ∂-incompressible: suppose A has a ∂-compression disk D. Since $A \cup K = T$ is a torus, D lies to one side of T, and so D is homotopic to a disk with ∂D either m or l. $\therefore$ ∂D intersects $\partial A = K$ either $|p|$ or $|q|$ times. But a ∂-compression disk intersects the boundary (i.e., K) in a single arc.

Definition: A surface F properly embedded in M^3 is essential if one of the following holds

(1) F is a 2-sphere that does not bound a ball
(2) F is a disk with $\partial F \subseteq \partial M$ not bounding a disk on ∂M
(3) F is not a disk or sphere, and is incompressible, ∂-incompressible, and not ∂-parallel.

Definition: A 3-manifold is
- irreducible if it contains no essential S^2
- boundary irreducible if it contains no essential D^2.
- atoroidal if it contains no essential torus
- anannular if it contains no essential annulus.

<u>Thm 8.13</u>: If M contains an embedded essential torus then M is not hyperbolic.

<u>proof</u>: If M contains an essential torus, then $\mathbb{Z} \times \mathbb{Z} \leq \pi_1(M)$.
$\therefore$ by a Corollary in Ch5, this $\mathbb{Z} \times \mathbb{Z}$ subgroup is generated by parabolics with a common fixed pt.
$\therefore$ by thick-thin decomposition, the torus is isotopic into a cusp, hence ∂-parallel.

<u>Corollary 8.14</u>: A satellite knot complement does not admit a hyperbolic structure.

<u>Theorem</u>: Suppose M^3 has torus boundary, and $\text{int}(M)$ has a complete, finite volume hyp-c metric. Then M cannot contain an essential annulus

<u>proof</u>: Suppose M is hyperbolic, and $A \subseteq M$ is an essential annulus with core curve γ.
Since $\partial A \subseteq \partial M$, γ is isotopic into ∂M.
$\therefore$ γ is isotopic into a cusp of $\text{int}(M)$, so it is isotopic to a curve of length $\to 0$
$\therefore$ $\text{hol}(\gamma)$ is a parabolic
since $\gamma \sim \partial A^+$ and $\gamma \sim \partial A^-$, both ∂-components of A correspond to the same parabolic elt.
$\therefore$ A is ∂-parallel.

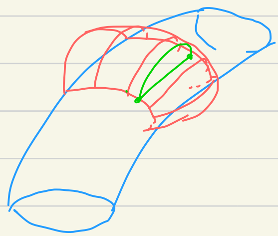
Corollary 8.16: A torus knot complement cannot admit a hyperbolic structure.

proof: If either p or q is ± 1, then $T(p,q)$ is the unknot, which is not hyperbolic.

The annulus $A = T \setminus K$ is ∂-incompressible (see Ex. 8.10) and incompressible (similar argument).

If A were ∂-parallel then it would be ∂-compressible, so it is not ∂-parallel.

$\therefore$ A is essential, so $T(p,q)$ is not hyperbolic.



Theorem 8.17 (Hyperbolization [Thurston '82, $\underbrace{\text{Kapovich '01}}_{\text{proof}}$]):

A knot complement is hyperbolic $\iff$ it is not a satellite knot or a torus knot.

More generally, a compact 3-mfld with nonempty torus boundary has interior admitting a finite volume (need to rule out $T^2 \times I$ and $\mathbb{T}$.) hyperbolic structure $\iff$ it is irreducible, ∂-irreducible, atoroidal, and anannular.

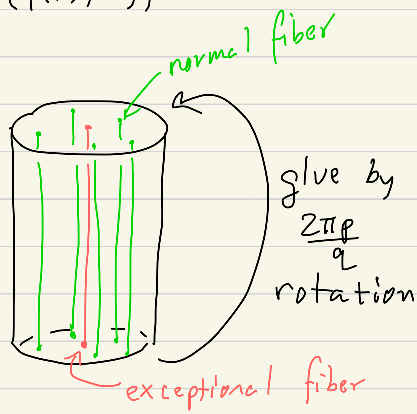
8.2: Torus decomposition, Seifert fibering, Geometrization

Definition: Let D^2 be the unit disk in $\mathbb{C}$, and let $f: D^2 \to D^2$ be the map $z \mapsto e^{2\pi p/q\, i} \cdot z$, $\gcd(p,q) = 1$.

A Seifert fibered solid torus of type (p,q) is obtained as the mapping torus

$$\mathbb{T} = D^2 \times I \big/ \{(x,0) \sim (f(x), 1)\}$$

- The fiber $\{0\} \times I$ is called the exceptional fiber

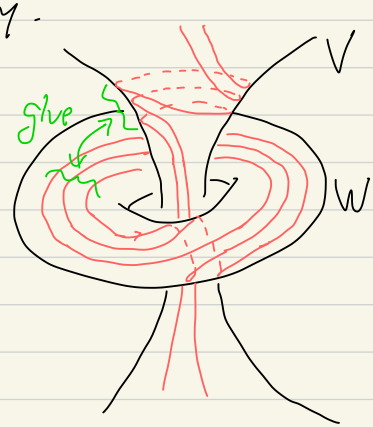


- All other fibers wrap around the solid torus q times, and are called normal fibers.

- If $q = 1$, call it a regularly fibered solid torus.

Definition: A Seifert fibered space is an orientable 3-mfld M that is the union of pairwise disjoint circles, called fibers, s.t. each fiber has a nbhd. diffeomorphic to a Seifert fibered solid torus.

Example 8.20: S^3 is the union of two solid tori V and W. For $(p,q)=1$, if V is SFST of type (p,q) and W is type (q,p), then gluing ∂V to ∂W identifies the fibers on the boundary.



Example 8.21: Remove a regular fiber on ∂V of the previous example. The result is (still) Seifert fibered, and is a (p,q) torus knot.

Theorem 8.22 (Casson–Jungreis '94, Gabai '92, others):

A compact, orientable irreducible 3-mfld M with infinite fundamental group is Seifert fibered $\iff$ $\pi_1(M)$ contains a normal infinite cyclic subgroup.

Remark: If $\Gamma \leq PSL_2\mathbb{C}$ is discrete torsion-free + non-elementary, then it has no normal inf. cyclic subgroup $\langle \varphi \rangle$, since φ and $g\varphi g^{-1}$ have different fixed points if φ and g do, so $g\varphi g^{-1} \neq \varphi^k$.

$\therefore$ $\mathbb{H}^3/\Gamma$ is Seifert fibered mfld.
$\iff$ Γ is elementary and torsion-free discrete.

Theorem 8.23 (JSJ decomposition [Jaco-Shalen, Johannson '79])

For any compact, irreducible, ∂-irreducible 3-mfld M, there exists a finite collection $\mathcal{T}$ of disjoint essential tori such that each component of $M \setminus \{\mathcal{T}\}$ is either atoroidal or Seifert fibered.

· A minimal such collection is unique (up to isotopy).

→ "the JSJ decomposition of M"

· union of Seifert fibered pieces is called the characteristic submanifold of M.

(which may be empty — e.g. Ex 8.4)

Theorem 8.25 (Geometrization for closed mflds).

Let M be a closed, orientable, irreducible 3-mfld,

Per02 Per03 (1) If $\pi_1(M)$ is finite, then M is spherical $\Rightarrow$ Poincaré conjecture
i.e., $M = S^3/\Gamma$, $\Gamma \leq O(4) \curvearrowright S^3$ without fixed pts.

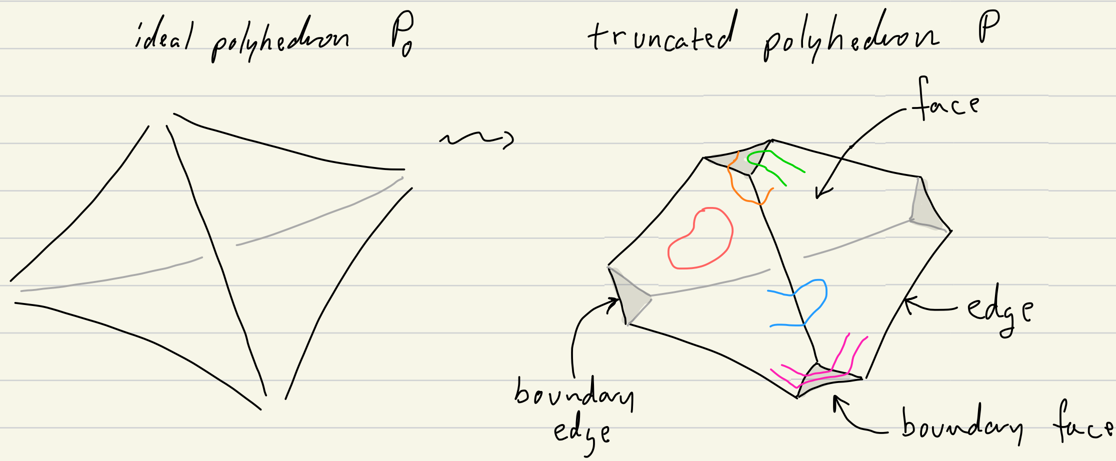
CJ94 + Gab92 (2) If $\pi_1(M)$ is infinite and contains a $\mathbb{Z}\times\mathbb{Z}$ subgroup, then M is either Seifert fibered or contains an incompressible torus (so not hyp-c)

Per02 Per03 (3) If $\pi_1(M)$ is infinite and contains no $\mathbb{Z}\times\mathbb{Z}$ subgroup, then M is hyperbolic.

8.3: Normal surfaces, angled polyhedra, and hyperbolicity

8.3.1: Normal Surfaces.

Definition 8.26:



- A properly embedded disk $(D, \partial D) \subseteq (P, \partial P)$ is normal if the following are satisfied:

(1) $\partial D \subseteq \partial P$ is transverse

(2) ∂D is not contained in a single face / ∂-face

(3) For F a face or ∂-face of P, the arc $\partial D \cap F$ does not have both ends on a single edge, or boundary edge, or an adjacent edge and ∂-edge.

(4) ∂D meets any edge at most once

(5) ∂D meets any ∂-face at most once

Definition 8.27: A surface is in normal form with respect to a polyhedral decomposition, or is normal, if it intersects the (truncated) polyhedra in normal disks.

Theorem 8.28 (Kneser '20, Haken '61, Shubert '61, others):

Suppose M admits an ideal polyhedral decomposition.
- If M contains an essential 2-sphere, then it contains one in normal form.
- If M is irreducible and M contains an essential disk, then it contains one in normal form.
- If M is irreducible + ∂-irreducible, and contains an essential surface, then that surface can be isotoped in M to be in normal form.

proof. Let $S \subseteq M$ be essential. Isotope S so that it is transverse to faces, ∂-faces, edges, and ∂-edges of the truncated polyhedra.
Let

$$f = |S \cap \{\text{faces}\}| + |S \cap \{\partial\text{-faces}\}|$$

$$e = |S \cap \{\text{edges}\}| + |S \cap \{\partial\text{-edges}\}|$$

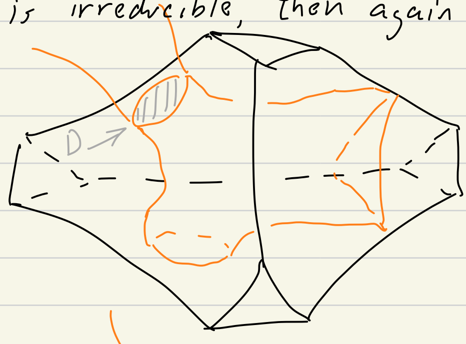
(f, e) is the complexity of S, and we order such tuples lexicographically.

Goal: adjust S to remove violations of (2)-(5), reducing it complexity at each step.
Finite complexity $\Rightarrow$ finitely many steps suffice.

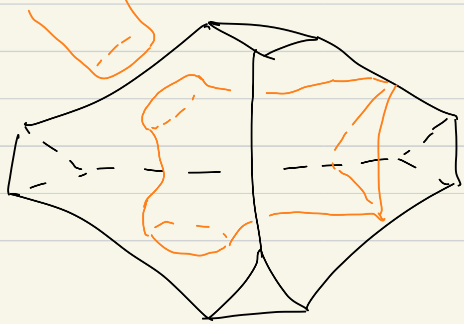
First: Can adjust S so that it intersects truncated polyhedra in disks, reducing complexity.

- suppose some component F of $S \cap P$ is not a disk. If S is a sphere, then F must be a sphere with holes (these are the only subsurfaces of S), so $F \cap \partial P$ is a union ≥ 2 circles. Each such circle bounds a disk D, and at least one component of $S \setminus D$ is incompressible. Surger S along D, pushing the resulting spheres away from F. Complexity is reduced.

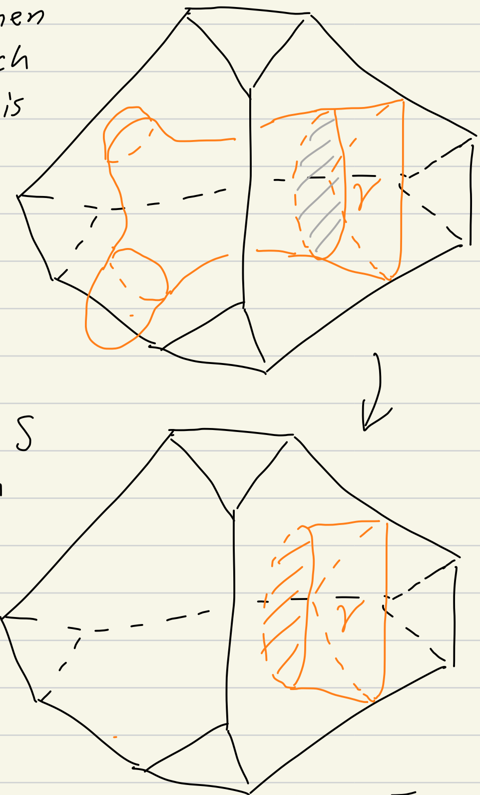
  If S is a disk and M is irreducible, then again F is an n-holed sphere, and a similar argument works.



  If M is irreducible and ∂-irreducible and S is essential: any curve γ of $S \cap \partial P$ bounds a disk on $\partial P \subseteq M$. In fact, by pushing γ into P along S, we get a disk $D \subseteq \operatorname{int}(P)$. Since S is essential, γ



bounds a disk D' on S. Then $D \cup D'$ is a sphere, which bounds a ball since M is irreducible. $\therefore$ we can isotope S across this sphere, removing the intersection with ∂P and reducing complexity.



$\therefore$ We may assume that S intersects the polyhedra in disk.

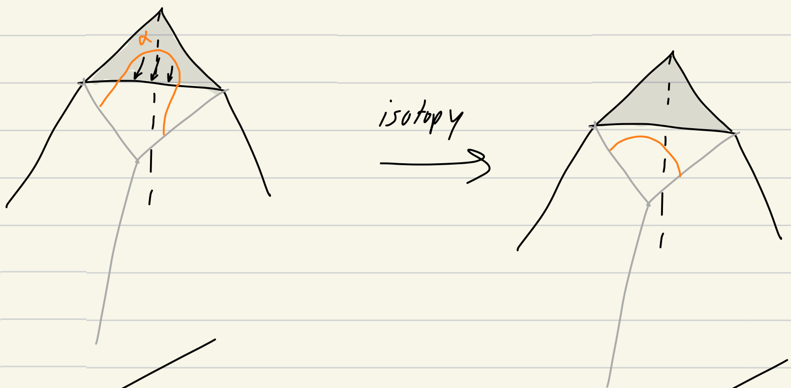
We now remove violations (2) - (5):

(2): If $\partial(S \cap P)$ contains a closed curve that is on a single face/∂-face F then there is an innermost such curve, which bounds a disk $D \subseteq F$.
If S is a 2-sphere, as before one component of S surgered along D is essential, and the surgery reduces complexity.
If M is irreducible and S is a disk, surger along D to get a disk and a sphere. M irreducible $\Rightarrow$ the sphere bounds a ball, so can isotope S across this ball, reducing complexity.

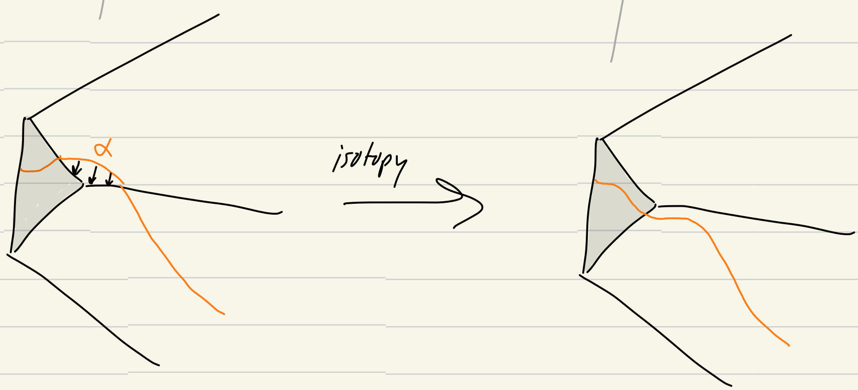
If M is irreducible and ∂-irreducible and S is essential, then ∂D bounds a disk $D' \subseteq S$, so $D \cup D'$ bounds a ball, and we can isotope S across this ball, reducing complexity.

(3): Suppose α is such an arc

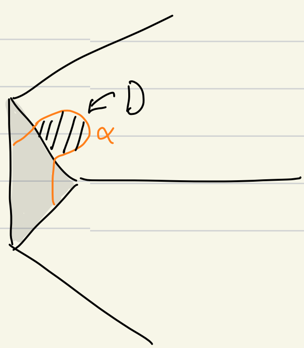
(a)



(b)



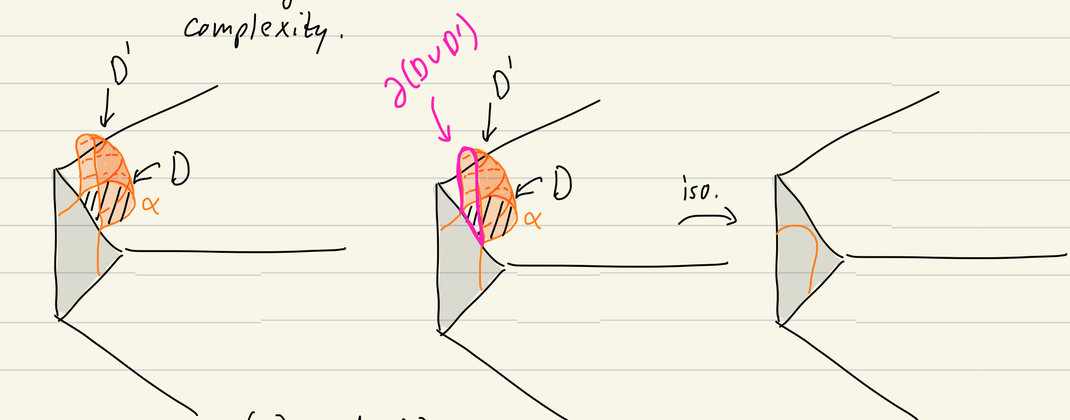
(c)



Cases (a) and (b) are removed with the isotopies shown, reducing complexity

For case (c): S is not a sphere (since it has bdy), so M is irreducible. If S is a disk, surger along the disk D shown, obtain two disks D' and D''. Both of these are of lower complexity, and one must be essential since S is, so replace D with the essential disk.

If S is not a disk, then M is irreducible + ∂-irreducible. Since S is essential D is not a ∂-compression disk, so the arc α bounds a disk $D' \subseteq S$ (with an arc on ∂S) $\therefore$ $D' \cup D$ is a disk with $\partial(D' \cup D) \subseteq \partial M$. Since M is ∂-irreducible, $\partial(D' \cup D)$ bounds a disk D'' in ∂M, and $D \cup D' \cup D''$ bounds a ball. By isotoping across the ball, we remove α and reduce complexity.

(c) 

(4) and (5): exercise.

8.3.2 Angle structures and combinatorial area

Recall that if a 3-mfld decomposes into ideal tetrahedra, then a complete hyperbolic structure satisfies edge equations and completeness equations.

Edge equations: $\underbrace{\prod_j z(e_j) = 1}_{\text{non-linear}}$ and $\underbrace{\sum \mathrm{Arg}(z(e_j)) = 2\pi}_{\text{linear}}$

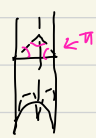
In general, solving the gluing equations is hard, but solving just the linear part of the edge equations is easy.

<u>Def-n 8.29</u>: An <u>angle structure</u> on an ideal triangulation T of a mfld M is a collection of dihedral angles for the tetrahedra edges s.t.

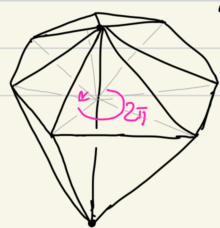
(0) Opposite edges of a tetrahedron have the same angle.

(1) Each dihedral angle is in $(0, \pi)$

(2) The same of the angles around an ideal vertex is π



(3) The sum of the angle around an edge of T is 2π



Denote the set of all angle structures on T by $A(T)$.

Remarks:

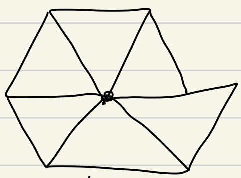
(3) is the linear part of the edge equations

(0) guarantees that the angles are compatible with a hyperbolic tetrahedon

(1) ensures the tetrahedra are not flat or degenerate

(2) ensures the triangular cross section at a vertex has a Euclidean structure.

- All equations coming from (0) → (3) are linear
- An angle structure on a tetrahedron determines a unique hyperbolic structure on the tetrahedron
- We have thrown out the non-linear part of the edge equations which prevents shearing singularities:



⇒ an angle structure does <u>not</u> necessarily give a hyperbolic structure (and if it does, the structure may not be complete, since we have not included completeness equations).

We can also assign dihedral angles to polyhedra. To generalize the notion of an angle structure to this setting we need:

Def-n 8.30: Let D be a normal disk in a (truncated) ideal polyhedral decomposition of M, such that each ideal edge of M has been assigned an interior angle in $(0,\pi)$. Let $\alpha_1, \dots, \alpha_n$ be the angles assigned to the ideal edges (non ∂-edges) met by ∂D. The combinatorial area of D is

$$a(D) = \sum_{i=1}^{n} (\pi - \alpha_i) - 2\pi + \pi \cdot \underbrace{|\partial D \cap \partial M|}_{\#\text{ of intersection with }\partial\text{-faces.}}$$

If S is a normal surface, then the combinatorial area of S is the sum of the comb-l areas of the normal disks of S.

- Note: If D is totally geodesic, and $D \perp \partial P$, then $a(D)$ is the hyperbolic area of D (exercise).

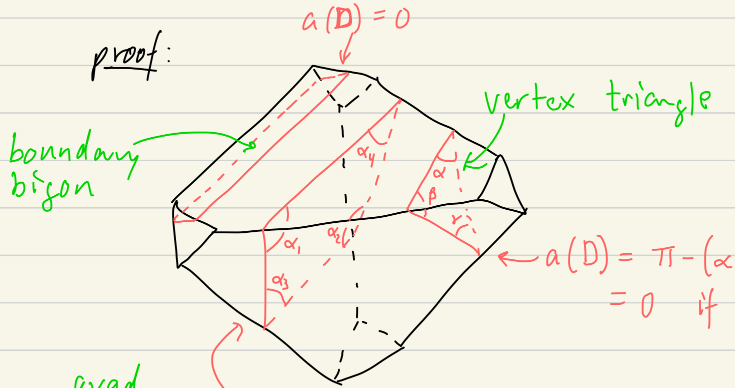
Def-n 8.31: An angled polyhedron structure on a 3-mfld M is a decomposition of M into ideal polyhedra, along with a collection of dihedral angles assigned to the edges of the polyhedra s.t.

(1) Each dihedral angle lies in $(0,\pi)$
(2) Every normal disk has non-negative combinatorial area.
(3) Interior angles around an edge sum to 2π

In particular, angle structures are examples of angled polyhedra structures:

Lemma 8.33: Let M be a triangulated 3-mfld with an angle structure. Then the combinatorial area of any normal disk D in a ideal tetrahedron of M is non-negative. It is zero $\iff$ D is a vertex triangle or a boundary bigon.

proof:



$\leftarrow a(D) = \pi - (\alpha+\beta+\gamma)$
$= 0$ if an angle structure

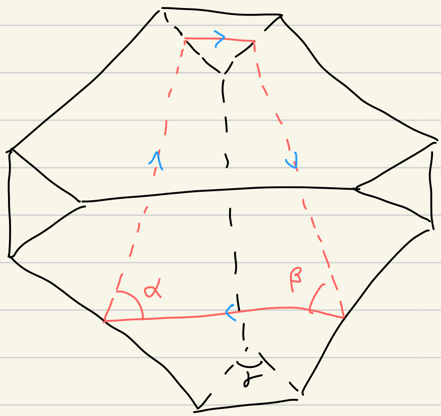
quad (meets no ∂-faces)

$$a(D) = 2\pi - (\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$
$$= 2(\pi - \alpha_1 - \alpha_3) > 0 \text{ if an angle structure}$$

$\hookrightarrow$ since $\alpha_1+\alpha_3 < \pi$

If D meets ≥ 2 ∂-faces, and at least one ideal edge (so not a ∂-bigon), the $a(D) \geq \sum(\pi - \alpha_i) > 0$.

- If D meets exactly 1 ∂-face, it must look like this:



$$\therefore\ a(D) = 2\pi - \alpha - \beta - 2\pi + \pi$$
$$= \pi - (\alpha + \beta) > 0$$
since $\alpha + \beta + \gamma = \pi$

- All other cases are covered in the picture on previous page.

$\square$

It follows that:

<u>Theorem</u> 8.34. An angle structure on an ideal triangulation is an angled polyhedral structure.

<u>Lemma</u> 8.35 (Gauss-Bonet): A normal surface S in an angled polyhedral structure satisfies

$$a(S) = -2\pi\chi(S)$$

proof: For a normal surface S, $\chi(S) = v - e + f$,

where $f = \#\{\text{normal disks}\}$

$e = \#$ of $\cap$'s of S with faces

$v = \#$ of $\cap$'s with ideal edges.

(intersections w/ ∂faces and ∂-edges do not contribute anything to $\chi(S)$).

$$a(S) = \sum_D a(D) = \sum_D \left(\sum_i (\pi - \alpha_i) + \pi(\partial D \cap \partial M) - 2\pi \right)$$

$$= \pi \sum_D \left(\left(\sum_i 1 \right) + |\partial D \cap \partial M| \right) - \overbrace{\sum_D \sum_i \alpha_i}^{2\pi v} - \overbrace{\sum_D 2\pi}^{-2\pi f}$$

Claim: $\left(\sum_i 1 \right) + |\partial D \cap \partial M| = \#\{\text{edges of } D\}$ (not ∂-edges)

Let D' be the result of shrinking ∂-edges of D to vertices. Then the number of vertices of D' is $\left(\sum_i 1\right) + |\partial D \cap \partial M|$, and the number of edges of D' is the same. Edges of D' and exactly the non-∂-edges of D.

Since each edge of S appears on 2 disks, we get

$$\pi \sum_D \left(|\partial D \cap \partial M| + \sum_i 1 \right) = 2\pi e$$

$\square$

angle poly'l str.
$\cancel{\Rightarrow}$ no.
$\Downarrow$?
angle structure

8.3.3 Hyperbolicity

<u>Theorem 8.36</u>: Let M be a mfld admitting an angled polyhedral structure. Then M is irreducible and ∂-irreducible.
Moreover, if the angled polyhedral structure is an angle structure, then M is atoroidal and annular and has torus boundary

TLDR; $\exists$ an angle structure $\Rightarrow \exists$ a hyp-c structure.

Converse?

M a hyp-c link complement $\Rightarrow \exists$ a triangulation with an angle structure

proof:

- irreducible: suppose S is an essential sphere. We can put S in normal form by Thm 8.28. Since normal disks have non-negative combinatorial area, $a(S) \geq 0$. On the other hand, by Gauss-Bonet, $a(S) = -4\pi$. Contradiction.

- ∂-irreducible: S an essential disk $\Rightarrow a(S) = -2\pi$, again contradicting $a(S) \geq 0$.

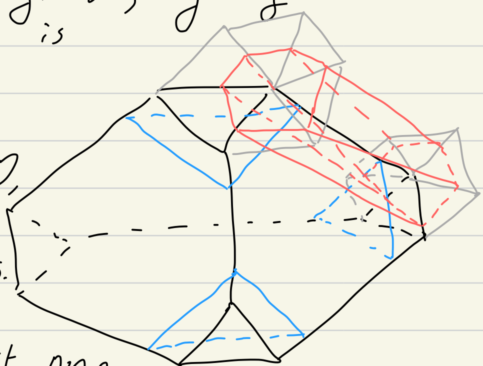
Now assume that the angled polyhedral structure is an angle structure

- torus boundary: boundary components come from gluing ∂-faces, which are isotopic to vertex triangles. Since $a(D) = 0$ for D a vertex triangle, $\chi(S) = 0$ for S a boundary component. $\therefore$ S is a torus.

- atoroidal: Let S be an essential torus. S can be put into normal form since M is irreducible + ∂-irreducible, and $a(S) = 0$ by Gauss-Bonnet.

  $\therefore$ each normal disk has combinatorial area 0, so is a vertex triangle or a boundary bigon. But S is closed, so boundary bigons do not appear.

  Vertex triangles can only glue to other triangles at the same vertex. To get a closed surface, all vertex triangles at a given vertex are glued, giving a ∂-parallel torus. $\therefore$ S is not essential.



- anannular: Again, $a(S)=0$, so S is made of vertex triangles and ∂-bigons. Since S has non-empty boundary, there is at least one ∂-bigon. But ∂-bigons only glue to other ∂-bigons at the same edge (class), and these glue up into a compressible annulus.

8.4 : Pleated Surfaces and a 6-theorem

Definition: An embedded surface $S \subseteq M$ is homotopically boundary incompressible if for any properly embedded arc α in S that is not homotopic rel endpoints into ∂S, the arc α in M is not homotopic rel endpoints into ∂M

- i.e., non-trivial arcs in S are non-trivial in M.
- $\Rightarrow$ ∂-incompressible

Lemma 8.38: Let M be a compact mfld s.t. $\operatorname{int}(M)$ is hyperbolic, and let $(S, \partial S) \subseteq (M, \partial M)$, $\partial S \neq \emptyset$, with S homotopically ∂-incompressible. Then the ideal edges of any triangulation of S can be homotoped to be geodesics in M. Similarly, each ideal triangle can be homotoped to be totally geodesic in M.

proof: Let e be an ideal edge of a triangulation of S. Since e is homotopically non-trivial in S, it is homotopically non-trivial in M. $\therefore$ can lift e to $\tilde{e} \subseteq \mathbb{H}^3$, and $\tilde{e}$ has distinct endpoints on $\partial \mathbb{H}^3$, so homotopic to a geodesic. This descends to a homotopy of e.
For a triangle t in an ideal triangulation of S, we first homotope the edges e_1, e_2, e_3 of t to geodesics in M.

Now, lift e_1 to a geodesic $\tilde{e}_1$ in $\mathbb{H}^3$. Since int(t) is simply connected it lifts to the interior of a triangle $\tilde{t}$ in $\mathbb{H}^3$ (determined by $\tilde{e}_1$).
The other two edges of $\tilde{t}$ are lifts $\tilde{e}_2$ and $\tilde{e}_3$ of e_2 and e_3. The $\tilde{e}_i$ bound a unique totally geodesic triangle that is homotopic to $\tilde{t}$.
Now push the homotopy down to M.

$\square$

The homotopy in the above proof is called straightening. Note that we can straighten all triangles of triangulation simultaneously, but the result may not be smooth or embedded

may be bent along edges.

Definition 8.39: A pleated surface in a hyp-c 3-mfld is a pair (S, φ) consisting of a surface S with complete hyp-c structure, and a local isometry $\varphi : S \to \varphi(S) \subseteq M$ such that each point of S lies in a geodesic mapped to a geodesic by φ.

Proposition 8.40: A homotopically ∂-incompressible surface S with $\partial S \neq \emptyset$ properly embedded in a hyperbolic 3-mfld can be pleated.

proof: straighten S in M (w.r.t. an ideal triangulation of S). The triangles of S can be realized as triangles in $\mathbb{H}^2$, so that they glue up into a fundamental domain for a hyperbolic surface S'. The map $\varphi: S' \longrightarrow S \subseteq M$ that maps the triangles back to S is the pleating map.

Let M be a 3-mfld with a torus boundary such that $int(M)$ is hyperbolic, and let T be the torus boundary of a cusp neighborhood for a cusp of M. Recall that an isotopy class of simple closed curves on T is called a slope.

Def-n 8.41: The slope length $l(s)$ of a slope s on T is the length of a geodesic reprentative of s with respect to the induced Euclidean metric on T.

Theorem 6.42 (A 6-theorem):

(Suppose M is a compact mfld with torus boundary, such that $\mathrm{int}(M)$ is hyperbolic. Let $s_1, \dots, s_n$ be slopes on distinct boundary components of M s.t. $\ell(s_i) > 6$ for all i with respect to Euclidean metrics coming from a disjoint collection of horospherical cusp tori for M.

Then the mfld $M_{(s_1,\dots,s_n)}$ coming from Dehn filling M along the s_i is) (irreducible, boundary irreducible, anannular, and atoroidal.)

Theorem (The 6-Theorem [Agol, Lackenby '00])

('') (hyperbolic)

Rmk: If $\partial(M_{(s_1,\dots,s_n)}) \neq \emptyset$, then Thm 6.42 give hyperbolic (by hyperbolization).

We will prove Thm 8.42, but not <u>the</u> 6-theorem.

plan: Assume M is reducible, ∂-reducible, annular, or toroidal, and show that there is then an essential punctured sphere or torus $S \subseteq M$ with $\partial S \subseteq \{s_1, \ldots, s_n\}$. Then pleat S and using the resulting metric to show that the s_i are at most 6.

<u>Lemma 8.43</u>: Let M, $s_1, \ldots, s_n$ be as in Thm 8.42. Suppose $M_{(s_1, \ldots, s_n)}$ contains an essential sphere, disk, annulus, or torus. Then M contains an essential, homotopically ∂-incompressible punctured sphere or torus S, with $\partial S \subseteq \{s_1, \ldots, s_n\}$, $\chi(S) \leq -1$.

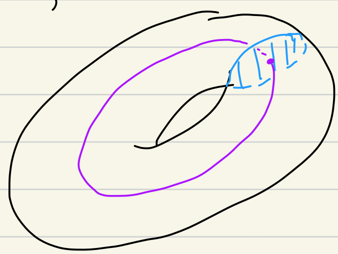
<u>proof</u>: Let $M_s = M_{(s_1, \ldots, s_n)}$. Let $F \subseteq M_s$ be an embedded essential sphere, disk, torus, or annulus. Note that $M \subset M_s = M \cup \{\text{solid tori}\}$.

If $F \subseteq M \subseteq M_s$, then since M is hyperbolic F is not essential in M. In this case F has a compression disk $D \subseteq M \subseteq M_s$, which is impossible since F is essential in M_s.

$\therefore$ F cannot be isotoped away from every filling solid tori.

For each filling solid torus T_i, we can either make F disjoint from T_i (by isotopy), or make F transverse to the core curve of T_i, with $F \cap \partial T_i = \left\{ \begin{matrix} \text{curves} \\ \text{isotopic to } s_i \end{matrix} \right\}$

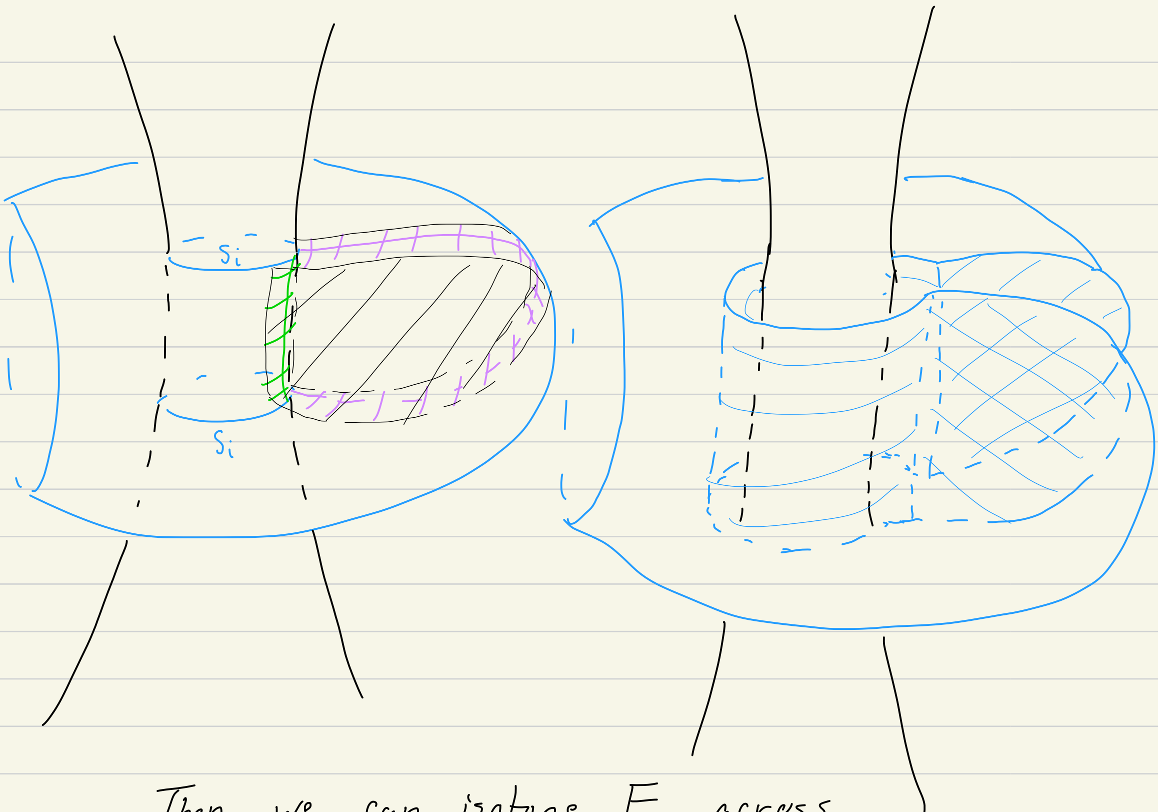
- If $F \cap \partial T_i \supseteq \alpha$ is a curve, then either α bounds a disk in T_i, or there is another curve $\alpha' \subseteq F \cap \partial T_i$ so that $\alpha \cup \alpha'$ bound an annulus $A \subseteq F \cap T_i$ (i.e., if F goes into T_i, it must be capped of by a disk in T_i, or come back out)



T_i

From the above, it follows that $S = F \cap M$ is a surface with $\partial S = \partial F \cup \{\gamma_1, \ldots, \gamma_k\}$, where each γ_j is isotopic to some s_i, and k is minimal (i.e., isotope F to intersect the T_i minimally).
Note that S cannot be compressible, since a compression disk for S would also be a compression disk for $F \supseteq S$.

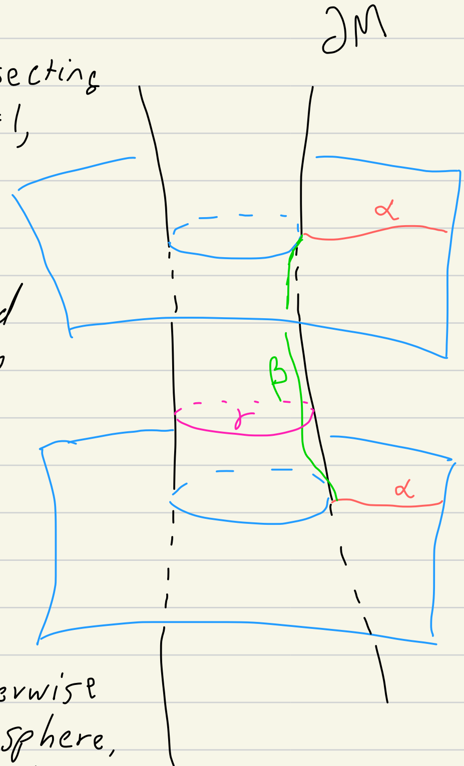
If D is a ∂-compression disk for S, then $\partial D = \alpha \cup \beta$, $\alpha \subset S$, $\beta \subseteq \partial T_i$ for some T_i.
Since $\partial D = \alpha \cup \beta$, β is isotopic in M to α, so we can isotope F across $D \times I$, so that $\beta \times I$ is taken to $\alpha \times I$.



Then we can isotope F across $\alpha \times D_i$, where D_i is the disk bounded by S_i (i.e., it is the meridional disk of the filling solid torus. This removes the intersection of F with T_i, contradicting minimality of $F \cap \{T_i\}$.

It remains to show that S is homotopically ∂-incompressible. If not, then there is a non-trivial arc $\alpha \subset S$ that is trivial in M, i.e., α is homotopic into ∂M. So α is homotopic to an arc $\beta \in \partial M$.

Thus $\alpha \cup \beta$ bounds a disk D in M, which may be immersed.
Consider $N = M \setminus S$.
Let $\gamma \subset \partial M$ be a curve isotopic to ∂S and intersecting β once. Since $|\gamma \cap (\alpha \cup \beta)| = 1$, $\alpha \cup \beta$ is non-trivial on ∂N.



Thus by the Loop Thm, $\alpha \cup \beta$ bounds an embedded disk in N, and hence also in M.
$\therefore$ $\alpha \cup \beta$ bounds a ∂-compression disk, which we have already shown is impossible.

Last, $\chi(S) \leq -1$ since otherwise S would be an essential sphere, disk, annulus, or torus, which is impossible since M is hyperbolic.

$\square$

Lemma 8.44: Let M be an orientable cusped hyp-c 3-mfld, and let C be a cusp neighborhood of a cusp, $T = \partial C$.

Let $f: S \to M$ be a pleating of a punctured surface S, with n of its punctures mapping to C. A loop about any of these punctures is represented by a geodesic on T. Let λ be the length of this geodesic (in T).

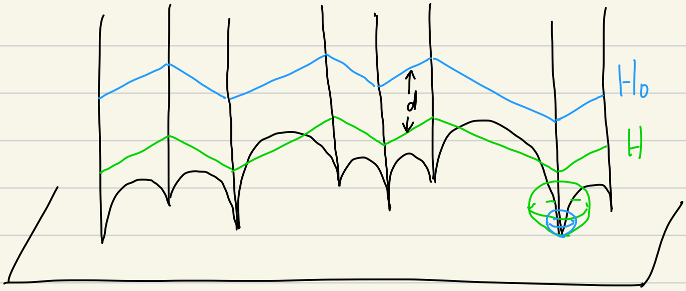
Then in the hyperbolic metric on S, $f^{-1}(C) \subseteq S$ contains horospherical cusp neighborhoods $R_1, \dots, R_n$ of the n punctures of S, with disjoint interiors, s.t.

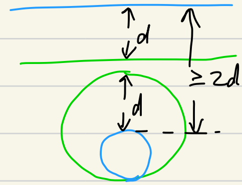
$$\ell(\partial R_i) = \text{area}(R_i) \geq \lambda \quad \forall\, i.$$

proof: Let τ be an ideal triangulation of S compatible with the pleating map.

Let $C_0 \subseteq C$ be a smaller cusp neighborhood, chosen so that $C_0 \cap f(\tau^{(1)})$ consists of arcs limiting into ideal vertices of τ. I.e., $f(S) \cap C_0$ consists of tips of triangles.

Then $f^{-1}(C_0)$ is a collection of embedded cusps $R_1^0, \dots, R_n^0$ in S.





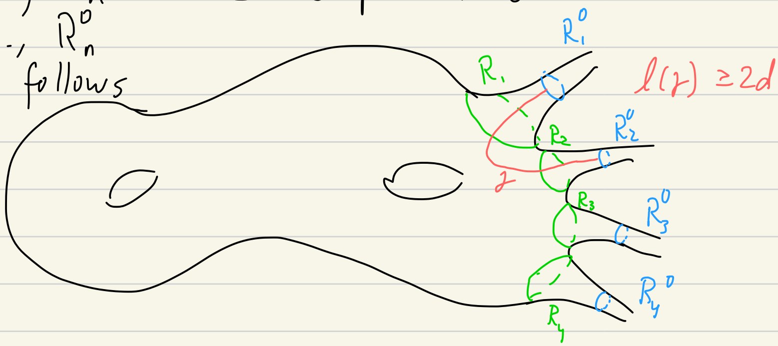
Let $\varphi : M \to \mathbb{H}^3$ be the universal cover.

C and C_0 both lift to disjoint collections of horoballs via φ. Since $C_0 \subseteq C$, for each $H \subseteq \varphi^{-1}(C)$, there is a horoball $H_0 \subseteq \varphi^{-1}(C_0)$, $H_0 \subseteq H$.

Let $d = dist_{\mathbb{H}^3}(H_0, H)$. Then the distance between any two lifts of C_0 must be at least $2d$.

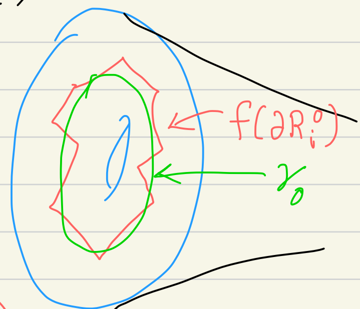
$\therefore$ The distance from R_i^0 to R_j^0 is at least $2d$ for $i \neq j$.

Taking $R_1, \ldots, R_n$ to be cusps in S distance d from $R_1^0, \ldots, R_n^0$ respectively, it follows that the R_i must be embedded.



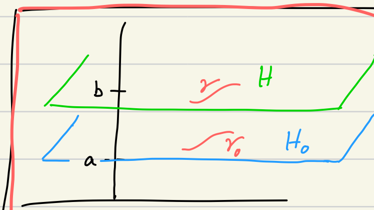
Now, let γ_0 be a Euclidean geodsic on ∂C_0 representing $f(\partial R_i^0)$.

Then $\ell(\gamma_0) \leq \ell(f(\partial R_i^0)) = \ell(\partial R_i^0)$



Since $\mathrm{dist}_M(\partial C, \partial C_0) = d$, if γ is a loop on ∂C homotopic to γ_0, then

$$\lambda = \ell(\gamma) = e^{-d}\ell(\gamma_0)$$



$d = \log(b/a) \rightsquigarrow b/a = e^d$

$\ell_{\mathbb{H}^2}(\gamma) = \ell_{\mathbb{E}^2}(\gamma)/b = \ell/b$

$\ell_{\mathbb{H}^2}(\gamma_0) = \ell_{\mathbb{E}^2}(\gamma_0)/a = \ell/a$

$\Rightarrow \ell_{\mathbb{H}^2}(\gamma) = \ell/a \cdot \frac{a}{b} = e^{-d}\ell_{\mathbb{H}^2}(\gamma_0)$

Similarly, $\ell(\partial R_i) = e^{-d}\ell(\partial R_i^0)$

$\therefore$

$$\lambda = \ell(\gamma) = e^{-d}\ell(\gamma_0) \leq e^{-d}\ell(\partial R_i^0)$$

$$= e^{-d} \cdot e^{d}\,\ell(\partial R_i) = \ell(\partial R_i)$$

Since $f(R_i^0) \subseteq C_0$, and $f(R_i)$ is contained in a d-nbhd. of C_0, we have that

$f(R_i)$ is contained in $C = d$-nbhd of C_0.

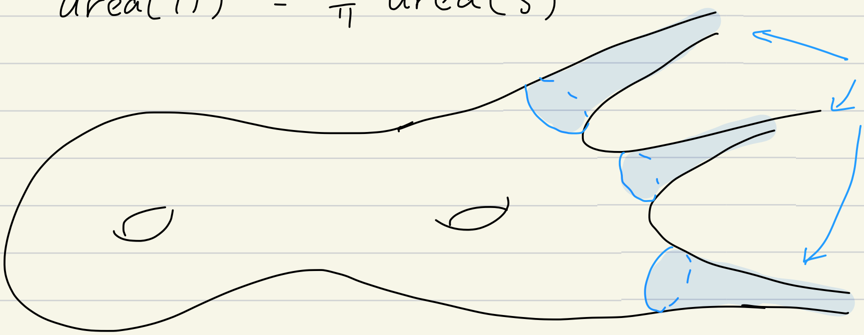
The fact that $\ell(\partial R_i) = \mathrm{area}(R_i)$ is an easy computation (exercise).

$\square$

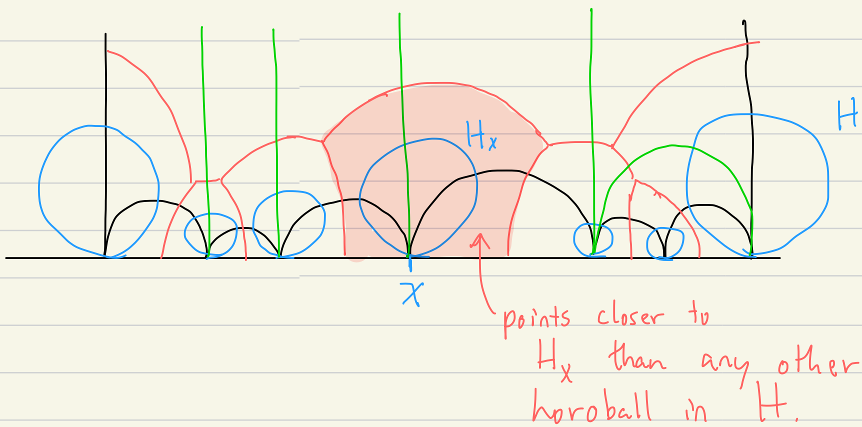
Theorem 8.45 (Böröczky cusp density thm):

Let S be a hyperbolic surface with cusps, and let H be an embedded horoball neighborhood for the cusps of S. Then

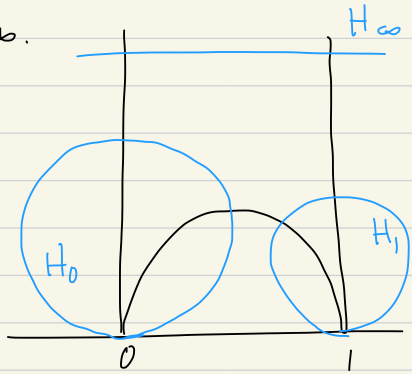
$$\operatorname{area}(H) \leq \frac{3}{\pi} \operatorname{area}(S)$$



proof: Given S and H, there is an ideal triangulation $\mathcal{T}$ of S s.t $H \cap \mathcal{T}$ consists only of corners of triangles:



Let T be a triangle in τ, and map to isometrically to a triangle $T' \subseteq \mathbb{H}^2$ with vertices $0, 1, \infty$. This maps sends horoballs of H intersecting T to H_0, H_1, H_∞.



$$\mathrm{area}(H \cap T)$$

$$= \mathrm{area}(H_0 \cap T') + \mathrm{area}(H_1 \cap T') + \mathrm{area}(H_\infty \cap T')$$

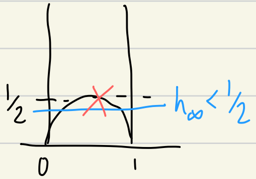
and $\quad \mathrm{area}(H) = \sum_{T \in \tau} \mathrm{area}(H \cap T)$

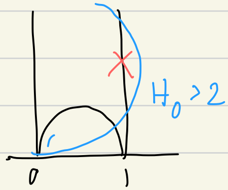
Since $\mathrm{area}(S) = \pi \cdot t$, where $t = \#$ of triangles,

$$\frac{\mathrm{area}(H)}{\mathrm{area}(S)} = \frac{1}{\pi} \underbrace{\frac{1}{t} \sum \mathrm{area}(H \cap T)}_{\text{average of the area}(H\cap T)} \leq \frac{1}{\pi} \cdot \max_{T \in \tau} \{\mathrm{area}(H \cap T)\}$$

So we need to maximize $\mathrm{area}(H \cap T)$.

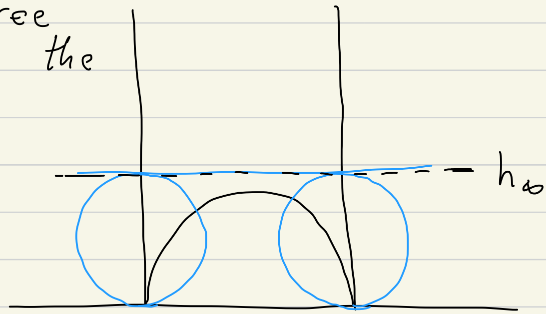
Now consider H_∞, H_0, H_1. Since $H_* \cap T'$ is a corner, $h_\infty = \mathrm{height}(H_\infty) \geq 1/2$, and $h_i = \mathrm{diam}(H_i) \leq 2$, $i = 0, 1$:





Since $\mathrm{area}(H \cap T)$ is clearly maximized when

each H_* is tangent another one (otherwise we could make one larger), it follows that one of the three horoballs is tangent to the other two.
W.l.o.g., we may assume that horoball is H_∞.



An easy computation gives

$$\operatorname{area}(H_\infty \cap T') = \frac{1}{h_\infty}$$

and

$$\operatorname{area}(H_0 \cap T') = \operatorname{area}(H_1 \cap T') = h_\infty,$$

so that

$$\operatorname{area}(H \cap T) = \frac{1}{h_\infty} + 2h_\infty.$$

This function is maximized for $h_\infty = 1$

Thus we have

$$\frac{\operatorname{area}(H)}{\operatorname{area}(T)} \leq \frac{1}{\pi} \cdot \max_{T \in \Upsilon} \{\operatorname{area}(H \cap T)\} = \frac{3}{\pi}$$

$\square$

proof of Thm 8.42:

Suppose that $M_S = M_{(s_1,\dots,s_n)}$ is reducible, ∂-reducible, annular, or toroidal.

Lemma 8.43 $\Rightarrow$ $\exists$ an embedded essential punctured 2-sphere or torus $S \subseteq M$.

Furthermore, if S is a torus, then each component of $\partial S \subseteq \partial M$ is parallel to some s_i. If S is a punctured sphere, then all but at most 2 boundary components of S are parallel to some s_i.

If M_s has an essential annulus or disk, then S has 1 or 2 ∂-components coming from the disk or annulus boundary.

Prop 8.40 $\Rightarrow$ S may be pleated.

Lemma 8.44 $\Rightarrow$ pleating induces horoball nbhds. $R_1, \dots, R_m$ of S s.t.

$$\ell(\partial R_i) = \text{area}(R_i) \geq \ell(s_{j_i})$$

where $f(\partial R_i)$ has slope s_{j_i}. Let $H = \bigcup R_i$

Then Thm. 8.45 + Gauss-Bonet gives

$$\sum \ell(S_{j_i}) \le \sum_i \ell(\partial R_i) = \text{area}(H) \le \frac{3}{\pi}\,\text{area}(S)$$
$$= \frac{3}{\pi}\cdot 2\pi\,|\chi(S)| = 6\,|\chi(S)|$$

If S is a punctured sphere, then at least $m-2$ of the m boundary components of S have $S_{i_j} \in \{S_1, \dots, S_n\}$, so

$$6(m-2) < \sum \ell(S_{j_i}) \le 6\cdot|\chi(S)| = 6\cdot(m-2)$$

since $\ell(S_{j_i}) > 6$ for each $S_{j_i} \in \{S_1, \dots, S_n\}$.

$\Longrightarrow$ Contradiction.

If S is a punctured torus, then all of the m boundary components of S are parallel to some S_j, so $\ell(S_{j_i}) > 6$ for all i.

$$\therefore\ 6m < \sum \ell(S_{j_i}) \le 6\cdot|\chi(S)| = 6m$$

$\Longrightarrow$ Contradiction. $\square$